RINGS AND MODULES CHARACTERIZED BY
OPPOSITES OF ABSOLUTE PURITY

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Introduction

Subpurity domain of a module

Rings whose simple modules are absolutely pure or t.f.b.s.

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t.f.b.s. modules over commutative rings

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An opposite notion of relative injectivity which is introduced by (Aydoğdu and López-Permouth), studied by many authors.

A module $M$ is said to be $A$-subinjective if for every extension $B$ of $A$ any homomorphism $\varphi : A \to M$ can be extended to a homomorphism $\phi : B \to M$ (see, Aydoğdu and López-Permouth). It is easy to see that $M$ is injective if and only if $M$ is $A$-subinjective for each module $A$. 
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Proposition

The following statements are equivalent for a right module $N$.

1. $N$ is absolutely pure.
2. $N \otimes M \to E(N) \otimes M$ is a monomorphism for each finitely presented left module $M$.
3. $N \otimes M \to E(N) \otimes M$ is a monomorphism for each left module $M$. 
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Definition

Given a right module $M$ and a left module $N$, $M$ is absolutely $N$-pure if for every right module $K$ with $M \leq K$ the map $i \otimes 1_N : M \otimes N \to K \otimes N$ is a monomorphism, where $i : M \to K$ is the inclusion map and $1_N$ is the identity map on $N$. The subpurity domain of a module $M_R$, $S(M)$, is defined to be the collection of all modules $R N$ such that $M$ is absolutely $N$-pure.

- A right module $M$ is absolutely pure if and only if $S(M) = R - MOD$.
- $S(M)$ consists of the class of left flat modules.
- A right $R$-module $M$ is called test for flatness by subpurity (t.f.b.s.) if $S(M_R)$ consists of only flat left $R$-modules.
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Example

The ring of integers \( \mathbb{Z} \) is t.f.b.s.

Proposition

\[
\bigcap_{M \in \text{MOD}_R} S(M) = \{ N \in R - \text{MOD} | N \text{ is flat} \}.
\]
Proposition

Every ring has a t.f.b.s. module.

Proposition

The following statements are equivalent for a ring $R$.

1. $R$ is von Neumann regular.
2. Every right $R$-module is t.f.b.s.
3. There exists a right absolutely pure t.f.b.s. $R$-module.
In order to investigate when the ring is t.f.b.s. as a right module over itself, we need the following definition.

**Definition**

A ring $R$ is called right $S$-ring if every finitely generated flat right ideal is projective.

**Theorem**

A ring $R$ is right t.f.b.s. and a right $S$-ring if and only if $R$ is right semihereditary.

**Corollary**

A commutative domain is Prüfer if and only if it is t.f.b.s.
Definition
A module $A$ is said to be a test module for injectivity by subinjectivity (or t.i.b.s) if whenever a module $M$ is $A$-subinjective implies $M$ is injective (see, Alizade, Büyükaşık and Er).

Proposition
If $N$ is right t.i.b.s., then $N$ is right t.f.b.s.

There are t.f.b.s. modules which are not t.i.b.s.

Example
Every semihereditary ring is a t.f.b.s. as a right module over itself. On the other hand, $R_R$ is t.i.b.s. if and only if $R$ is right hereditary and right Noetherian (see, Alizade, Büyükaşık, and Er).
In searching the converse of above Proposition, we have the following.

**Proposition**

Let $R$ be a right Noetherian ring. If $M$ is a t.f.b.s. right $R$-module and $E(M)$ is finitely generated, then $M$ is right t.i.b.s.

**Proposition**

The following are equivalent for a ring $R$.

1. $R_R$ is t.f.b.s. and Noetherian.
2. $R_R$ is t.i.b.s.
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Theorem

Let $R$ be a right Noetherian ring. Every simple right module is t.f.b.s. or absolutely-pure (=injective) if and only if

1. $R$ is a right V-ring; or
2. $R \cong A \times B$, where $A$ is right Artinian with a unique non-injective simple right module and $\text{Soc}(A_A)$ is homogeneous and $B$ is semisimple.
### Proposition

Let $R$ be a right Noetherian ring. Every simple right module is t.f.b.s. or absolutely pure if and only if every simple right module is t.i.b.s. or injective.

### Proposition

Let $R$ be an arbitrary ring. Suppose that every simple module is t.i.b.s. or injective. Then $R$ is a right $V$-ring or right Noetherian.

### Corollary

The following are equivalent for a ring $R$.

1. Every simple module is t.i.b.s. or injective.
2. $(i)$ $R$ is a right $V$-ring, or
   $(ii)$ $R \cong A \times B$, where $A$ is right Artinian with a unique non-injective simple right $R$-module and $\text{Soc}(A_A)$ is homogeneous and $B$ is semisimple.
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Theorem

Let $R$ be a right Noetherian ring. Suppose that every right $R$-module is t.f.b.s. or absolutely pure. Then $R \cong A \times B$, where $B$ is semisimple, and

1. $A$ is right hereditary right Artinian serial with homogeneous socle, $J(A)^2 = 0$ and $A$ has a unique noninjective simple right $A$-module, or;

2. $A$ is a QF-ring that is isomorphic to a matrix ring over a local ring, or;

3. $A$ is right SI with $Soc(A_A) = 0$. 

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Proposition

The following are equivalent for a commutative domain $R$.

1. $R$ is Prüfer.
2. $R$ is t.f.b.s.
3. Every nonzero finitely generated ideal is t.f.b.s.
4. A finitely generated $R$-module $M$ is t.f.b.s. when $\text{Hom}(M, R) \neq 0$. 
Now we shall give a characterization of t.f.b.s. modules over commutative hereditary Noetherian rings. We begin with the following.

**Theorem**

Let $R$ be a commutative hereditary Noetherian ring and $F$ a flat module. Then $F$ is t.f.b.s. if and only if $\text{Hom}(F, S) \neq 0$ for each singular simple $R$-module $S$. 
Theorem

Let $R$ be a commutative hereditary Noetherian ring and $N$ be an $R$-module. The following are equivalent.

1. $N$ is t.f.b.s.
2. $N/Z(N)$ is t.f.b.s.
3. $\text{Hom}(N/Z(N), S) \neq 0$ for every singular simple $R$-module $S$.
4. $N/Z(N) \otimes S \neq 0$ for every singular simple $R$-module $S$. 
Corollary

Let $R$ be a Principal Ideal Domain. Then an $R$-module $G$ is t.f.b.s. if and only if $G/\mathcal{T}(G) \neq p(G/\mathcal{T}(G))$ for every irreducible element $p$ in $R$.

Corollary

Let $G$ be a finitely generated abelian group. Then the following are equivalent.

1. $G$ is t.f.b.s.
2. $G$ is t.i.b.s.
3. $\mathcal{T}(G) \neq G$. 
