Introduction to Space-Time Coding

Frédérique Oggier
frederique@ntu.edu.sg

Division of Mathematical Sciences
Nanyang Technological University, Singapore

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Last Time

- 1. A fully diverse space-time code is a family $\mathcal{C}$ of (square) complex matrices such that $\det(\mathbf{X} - \mathbf{X}') \neq 0$ when $\mathbf{X} \neq \mathbf{X}'$.
  2. Division algebras whose elements can be represented as matrices satisfy full diversity by definition.

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  2. Division algebras whose elements can be represented as matrices satisfy full diversity by definition.
- 1. For coding for MIMO slow fading channels, joint design of an inner and outer code.
  2. The outer code is a coset code, which addresses the problem of codes over matrices.
  3. Connection between codes over matrices and codes over finite fields.

Outline

Construction A

The non-commutative case

Non-associative Algebras

Construction A

- Let $\rho : \mathbb{Z}^N \mapsto \mathbb{F}_2^N$ be the reduction modulo 2 componentwise.
- Let $C \subset \mathbb{F}_2^N$ be an (N, k) linear binary code.
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What about a Construction A from division algebras?

Ingredients

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A & \Lambda \supset \mathfrak{p}\Lambda \\
| & \\
K & \mathcal{O}_K \supset \mathfrak{p}\mathcal{O}_K \\
\langle\sigma\rangle \,| & \\
F & \mathcal{O}_F \supset \mathfrak{p} \\
| & \\
\mathbb{Q} & \mathbb{Z} \supset p
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- Let K/F be a cyclic number field extension of degree n, and rings of integers $\mathcal{O}_K$ and $\mathcal{O}_F$. Consider the cyclic division algebra

$$\mathcal{A} = K \oplus Ke \oplus \cdots Ke^{n-1}$$

where $e^n = u \in \mathcal{O}_F$, and $ek = \sigma(k)e$ for $k \in K$.

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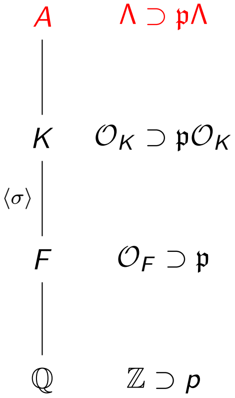

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- Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ so that $\mathfrak{p}\Lambda$ is a two-sided ideal of Λ.

Skew-polynomial Rings

- Given a ring S with a group $\langle \sigma \rangle$ acting on it, the skew-polynomial ring $S[x; \sigma]$ is the set of polynomials $s_0 + s_1 x + \ldots + s_n x^n$, $s_i \in S$ for $i = 0, \ldots, n$, with $xs = \sigma(s)x$ for all $s \in S$.

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- **Lemma.** There is an $\mathbb{F}_{p^f}$-algebra isomorphism between $\Lambda/\mathfrak{p}\Lambda$ and the quotient of $(\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)[x;\sigma]$ by the two-sided ideal generated by $x^n - u$.

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- There is an $\mathbb{F}_{p^f}$-algebra isomorphism

$$\psi : \Lambda/\mathfrak{p}\Lambda \cong (\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)[x;\sigma]/(x^n - u).$$

- If $\mathfrak{p}$ is inert, $\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K$ is a finite field

Codes over Finite Fields

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- Let $\mathcal{I}$ be a left ideal of Λ, $\mathcal{I} \cap \mathcal{O}_F \supset \mathfrak{p}$. Then $\mathcal{I}/\mathfrak{p}\Lambda$ is an ideal of $\Lambda/\mathfrak{p}\Lambda$ and $\psi(\mathcal{I}/\mathfrak{p}\Lambda)$ a left ideal of $\mathbb{F}_q[x;\sigma]/(x^n - u)$.

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- Let $f \in \mathbb{F}_q[x;\sigma]$ be a polynomial of degree n. If (f) is a two-sided ideal of $\mathbb{F}_q[x;\sigma]$, then a *σ-code* consists of codewords $a = (a_0, a_1, \ldots, a_{n-1})$, where $a(x)$ are left multiples of a right divisor g of f.

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- Using $\psi : \Lambda/\mathfrak{p}\Lambda \cong \mathbb{F}_q[x;\sigma]/(x^n - u)$, for every left ideal $\mathcal{I}$ of Λ, we get a σ-code $C = \psi(\mathcal{I}/\mathfrak{p}\Lambda)$ over $\mathbb{F}_q$.

[D. Boucher and F. Ulmer, Coding with skew polynomial rings]

Codes over Finite Rings

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- A parity check polynomial is computed.
- A dual code is defined.

[Ducoat-O., On Skew Polynomial Codes and Lattices from Quotients of Cyclic Division Algebras]

Lattices

$\Lambda/\mathfrak{p}\Lambda \, (\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)^n \supset C$

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- Then L is a lattice, that is a $\mathbb{Z}$-module of rank $n^2[F:\mathbb{Q}]$.

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- Taking the pre-image by ψ, it corresponds to the left-ideal $\mathcal{I}/3\Lambda$, with $\mathcal{I} = \Lambda(1 + i + e)$.

Example (II)

- For $q = a + be$ in $\mathbb{Z}[i] \oplus \mathbb{Z}[i]e \subset \mathfrak{Q}$, $a, b \in \mathbb{Z}[i]$

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where $\bar{\cdot}$ is the non-trivial Galois automorphism of $\mathbb{Q}(i)/\mathbb{Q}$.

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- Let $t = (a + be)(1 + i + e)$ be an element of $\mathcal{I} = \Lambda(1 + i + e)$. Then

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- Then $\mathcal{I} = \rho^{-1}(C)$ is a real lattice of rank 4 embedded in $\mathbb{R}^8$.

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- Coset encoding is necessary for wiretap codes: information symbols are mapped to a codeword in C, while random symbols are picked uniformly at random in the lattice $\mathfrak{p}\Lambda$ to confuse the eavesdropper.
- The lattice $L = \rho^{-1}(C) = \mathcal{I}$ thus enables coset encoding for wiretap space-time codes.

Summary

- Cyclic division algebras are useful for space-time coding. Some applications require to understand quotients of cyclic division algebras.
- The view point of skew-polynomial rings.
- Construction A of lattices from codes over skew-polynomial rings.
- Further work:
    1. Study the lattice properties inherited from codes.
    2. Study the space-time codes obtained.
    3. Study constacyclic codes over $(\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K)[x;\sigma]/(f(x))$, and duality with respect to a Hermitian inner product.

Construction A

The non-commutative case

Non-associative Algebras

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- Similar to associative quaternions, but for $\gamma \in K \backslash F$, which makes the multiplication not associative anymore.

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- Similar to associative quaternions, but for $\gamma \in K \backslash F$, which makes the multiplication not associative anymore.
- The algebra A is called a *non-associative quaternion algebra* over F. It is a division algebra.

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- Take $K \subset \mathcal{N}_r(A) \cap \mathcal{N}_l(A)$ or $K \subset \mathcal{N}_m(A) \cap \mathcal{N}_l(A)$, which is maximal with respect to inclusion. Consider A as a right K-vector space. We get an embedding

$$\lambda : A \to \mathrm{Mat}_r(K), \ a \mapsto \lambda_a$$

of vector spaces, $r = \dim_K(A)$.

An Example of Non-associative codebook

- Take $K = F(\sqrt{a}) = F(i)$, $\gamma \in K \backslash F$, and A a nonassocative quaternion divison algebras. Set $j = (0, 1)$. Then A has F-basis $\{1, i, j, ji\}$ such that $i^2 = a$, $j^2 = b$ and $xj = j\sigma(x)$ for all $x \in K$.

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- Consider the K-basis $\{1, j\}$ of A. We have an embedding $\lambda : A \to \mathrm{Mat}_2(K)$ which sends $x \in A$ to the matrix of λ_x in the basis $\{1, j\}$.
- This gives the codebook

$$\left\{ \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix}, \ x_0, x_1 \in K \right\}.$$

[S. Pumplün, T. Unger, "Space-Time Block Codes from Nonassociative Division Algebras."]

Take Home Message (I)

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2. Good space-time codes = *codes with full diversity*, can be obtained as multiplication matrices coming from cyclic division algebras.
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4. Recent constructions using *cyclic*, *crossed-products*, *non-associative* algebras.

Take Home Message (II)

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2. *Construction A* for space-time codes.

Open Questions

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1. Space-time block code modulation: characterization of quotients, weights and codes.
2. Construction A: lattices, space-time codes, constacyclic codes.

Thank you for your attention!