Neat Homomorphisms over Dedekind Domains

Salahattin ÖZDEMİR
(Joint work with Engin Mermut)

Dokuz Eylül University, Izmir-Turkey

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Neat Homomorphisms over Dedekind Domains

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Let $\mathcal{P}$ be a class of s.e.s of $R$-modules. If $E : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ belongs to $\mathcal{P}$, then $f$ is called a $\mathcal{P}$-monom., and $g$ is called a $\mathcal{P}$-epim.
Let $\mathcal{P}$ be a class of s.e.s of $R$-modules. If $\mathbb{E} : 0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$ belongs to $\mathcal{P}$, then $f$ is called a $\mathcal{P}$-monom., and $g$ is called a $\mathcal{P}$-epim.

**Proper class**

$\mathcal{P}$ is said to be **proper** if

- **P1.** If $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every s.e.s. isomorphic to $\mathbb{E}$.
- **P2.** $\mathcal{P}$ contains all splitting s.e.s.
- **P3.** The composite of two $\mathcal{P}$-monom. (resp. $\mathcal{P}$-epim.) is a $\mathcal{P}$-monom. (resp. $\mathcal{P}$-epim.) if these composites are defined.
- **P4.** If $g$ and $f$ are monom., and $gf$ is a $\mathcal{P}$-monom., then $f$ is a $\mathcal{P}$-monom. Moreover, if $g$ and $f$ are epim. and $gf$ is a $\mathcal{P}$-epim., then $g$ is a $\mathcal{P}$-epim.
Example of proper classes

1. $\text{Split}_R$: The smallest proper class of modules consists of only splitting s.e.s.

$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of abelian groups s.t. $\text{Im}f$ is a pure subgroup of $B$, where a subgroup $A \subseteq B$ is called pure in $B$ if $A \cap nB = nA$ for all integers $n$. 
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Let $\mathcal{M}$ be a class of modules.

proj., inj., flatly generated proper classes

- $\pi^{-1}(\mathcal{M})$: the class of all s.e.s. $E$ s.t. $\text{Hom}_R(L, E)$ is exact $\forall L \in \mathcal{M}$ (proper class proj. gen. by $\mathcal{M}$);
- $\iota^{-1}(\mathcal{M})$: the class of all s.e.s. $E$ s.t. $\text{Hom}_R(E, L)$ is exact $\forall L \in \mathcal{M}$ (proper class inj. gen. by $\mathcal{M}$);
- $\tau^{-1}(\mathcal{M})$: the class of all s.e.s. $E$ s.t. $L \otimes E$ is exact $\forall L \in \mathcal{M}$ (proper class flatly gen. by the class $\mathcal{M}$ of right modules).
A subgroup $A \subseteq B$ is called neat in $B$ if $A \cap pB = pA$ for all primes $p$ [Honda(1956)].

The class of all neat-exact sequences of abelian groups forms a proper class, denoted $\mathcal{N}eat\mathbb{Z}$. 
A subgroup $A \subseteq B$ is called neat in $B$ if $A \cap pB = pA$ for all primes $p$ [Honda(1956)].
The class of all neat-exact sequences of abelian groups forms a proper class, denoted $\mathcal{N}eat_\mathbb{Z}$.

- A is closed in $B$ if $A$ has no proper essential extension in $B$.

$Closed_\mathbb{Z} = \mathcal{N}eat_\mathbb{Z}$

The proper class $Closed_\mathbb{Z} = \mathcal{N}eat_\mathbb{Z}$ is proj., inj. and flatly generated by all simple groups $\mathbb{Z}/p\mathbb{Z}$ ($p$ prime number):

\[
Closed_\mathbb{Z} = \mathcal{N}eat_\mathbb{Z} \\
= \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\
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Neat submodules [Stenstrom(1967)]

A monomorphism $f : A \rightarrow B$ is called neat if any simple module $S$ is projective w.r.t $B \rightarrow B/\text{Im } f$.
A monomorphism \( f : A \to B \) is called neat if any simple module \( S \) is projective w.r.t \( B \to B/\text{Im } f \):

\[
\begin{array}{ccc}
S & \to & B/\text{Im } f \\
\downarrow & & \downarrow \\
B & \to & 0
\end{array}
\]
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$$\begin{array}{c}
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\downarrow \\
B \longrightarrow B/\text{Im } f \longrightarrow 0
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$A \subseteq B$ is called a neat submodule of $B$ if the inclusion $A \hookrightarrow B$ is neat.
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\[ \mathcal{N}eat_R \]

The class of all s.e.s. \( 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \) s.t. \( \text{Im } f \) is a neat submodule of \( B \) forms a proper class:

\[ \mathcal{N}eat_R = \pi^{-1}(\{ \text{all simple } R\text{-modules}\}) \]
Torsion free covers of modules [Enochs(1971)]

Over a commutative domain $R$, a homomorphism $\varphi : T \to M$, where $T$ is a torsion free $R$-module, is called a torsion free cover of $M$ if

(i) for every torsion free $R$-module $G$ and a homomorphism $f : G \to M$ there is a homomorphism $g : G \to T$ s.t. $\varphi g = f$:

\[
\begin{array}{ccc}
G & \xrightarrow{f} & M \\
\downarrow{g} & & \\
T & \xrightarrow{\varphi} & M
\end{array}
\]
Neat Homomorphisms over Dedekind Domains

Neat homomorphisms

Over a commutative domain \( R \), a homomorphism \( \varphi : T \rightarrow M \), where \( T \) is a torsion free \( R \)-module, is called a torsion free cover of \( M \) if

(i) for every torsion free \( R \)-module \( G \) and a homomorphism \( f : G \rightarrow M \) there is a homomorphism \( g : G \rightarrow T \) s.t. \( \varphi g = f \):

\[
\begin{array}{ccc}
G & \xrightarrow{g} & T \\
\downarrow & & \downarrow \varphi \\
\downarrow f & & \\
M & & M
\end{array}
\]

(ii) \( \text{Ker} \varphi \) contains no non-trivial submodule \( S \) of \( T \) s.t. \( rS = rT \cap S \) for all \( r \in R \) (i.e. \( S \) is an \textit{RD}-submodule (relatively divisible submodule) of \( T \)).
Neat homomorphisms of Enochs and Bowe

A homomorphism \( f : M \to N \) is neat if given any proper submodule \( H \) of \( G \) and any homomorphism \( \sigma : H \to M \), the homomorphism \( f \circ \sigma \) has a proper extension in \( G \) iff \( \sigma \) has a proper extension in \( G \).
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A homomorphism $f : M \rightarrow N$ is neat if given any proper submodule $H$ of $G$ and any homomorphism $\sigma : H \rightarrow M$, the homomorphism $f \circ \sigma$ has a proper extension in $G$ iff $\sigma$ has a proper extension in $G$. That is, a commutative diagram (1):

\[
\begin{array}{ccc}
H & \rightarrow & G' \\
\sigma \downarrow & & f \downarrow \\
M & \rightarrow & N
\end{array}
\]

always guarantees the existence of a commutative diagram (2):

\[
\begin{array}{ccc}
H & \rightarrow & G'' \\
\sigma \downarrow & & f \downarrow \\
M & \rightarrow & N
\end{array}
\]
Remark

These neat homomorphisms need not be monic or epic. It is different from our definition of neat, so we call them $E$-neat homomorphisms.
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Examples

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2. $A$ is a neat subgroup of $B$ iff the monom. $A \hookrightarrow B$ is $E$-neat.
3. $A$ is a closed submodule of $B$ iff the monom. $A \hookrightarrow B$ is $E$-neat (i.e. $A$ is $E$-neat submodule).
C-ring of Renault (1964)

$R$ is called a left $C$-ring if $\text{Soc}(R/I) \neq 0$, for every essential proper left ideal $I$ of $R$. 

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**neat = $E$-neat**

- $R$ is a left $C$-ring iff $\text{Closed} = \mathcal{N}eat_R$. [Generolov (1978)]
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- $R$ is a left C-ring iff $\text{Closed} = \mathcal{N}eat_R$. [Generolov (1978)]
- So, over a left C-ring, neat submodules and $E$-neat submodules coincide.
Theorem [Bowe(1972)]
Let \( f : A \to B \) be a homom. of modules. TFAE:

1. \( f \) is \( E \)-neat.
2. In the defn. of \( E \)-neat homom., it suffices to take \( G = R \) and \( H \) a left ideal of \( R \).
3. In the defn. of \( E \)-neat homom., it suffices to take \( \sigma \) a monom. and \( G \) as an essential extension of \( H \).
4. There are no proper extensions of \( f \) in the injective envelope \( E(A) \) of \( A \).
We usually use the characterization (4) for $E$-neat homom.

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\[
\begin{array}{ccc}
E(A) & & \\
\uparrow & & \\
L & & \\
\uparrow & & \\
A & f & B
\end{array}
\]

commutates, then $A = L$. 
Neat Homomorphisms over Dedekind Domains

The class of neat epimorphisms

• Since a monomorphism $f : A \to B$ is $E$-neat iff $\text{Im } f$ is a closed submodule, the class of all s.e.s. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ s.t. $f : A \to B$ is $E$-neat forms the proper class $\text{Closed}$. 
Since a monomorphism $f : A \to B$ is $E$-neat iff $\text{Im} \, f$ is a closed submodule, the class of all s.e.s. $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ s.t. $f : A \to B$ is $E$-neat forms the proper class $Closed$.

So, we investigate the class of all such s.e.s. s.t. $g : B \to C$ is $E$-neat, denoted by $ENeat$. 

The projection epimorphism $f : A \oplus B \to A$ is $E$-neat iff $\text{Ker} \, f \cong B$ is injective.

$ENeat$ is not proper in general. The class $ENeat$ satisfies the conditions $P1$ and $P3$ for every ring $R$, and $P4$ for left hereditary rings, but it does not satisfy the condition $P2$ unless the ring $R$ is semisimple. Thus, $ENeat$ forms a proper class if and only if $R$ is semisimple.
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**Theorem**

The projection epimorphism $f : A \oplus B \to A$ is $E$-neat iff $\text{Ker} f \cong B$ is injective.
- Since a monomorphism \( f : A \rightarrow B \) is \( E \)-neat iff \( \text{Im} f \) is a closed submodule, the class of all s.e.s. \( 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \) s.t. \( f : A \rightarrow B \) is \( E \)-neat forms the proper class \( \text{Closed} \).
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The class \( \mathcal{ENeat} \) satisfies the conditions P1 and P3 for every ring \( R \), and P4 for left hereditary rings, but it does not satisfy the condition P2 unless the ring \( R \) is semisimple.
• Since a monomorphism \( f : A \to B \) is \( E \)-neat iff \( \text{Im} \, f \) is a closed submodule, the class of all s.e.s. \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) s.t. \( f : A \to B \) is \( E \)-neat forms the proper class \( \text{Closed} \).
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The class \( \mathcal{ENeat} \) satisfies the conditions P1 and P3 for every ring \( R \), and P4 for left hereditary rings, but it does not satisfy the condition P2 unless the ring \( R \) is semisimple. Thus, \( \mathcal{ENeat} \) forms a proper class if and only if \( R \) is semisimple.
A homomorphisms $f : A \rightarrow B$ of modules is $E$-neat if for every decomposition $f = \beta \alpha$ where $\alpha$ is an essential monomorphism, $\alpha$ is an isomorphism:

\[ \begin{array}{c}
L \\
\downarrow \alpha \\
A \\
\downarrow f \\
\alpha \quad \beta \\
\downarrow \\
B
\end{array} \]
Theorem [Zoschinger(1978)]

Let $f : A \rightarrow B$ be a homom. of abelian groups. TFAE:

1. $f$ is $E$-neat;
2. $\text{Im } f$ is closed in $B$ and $\text{Ker } f \subseteq \text{Rad } A$;
3. $f^{-1}(pB) = pA$ for all prime numbers $p$;
4. if the following diagram is pushout of $f$ and $\alpha$, and $\alpha$ is an essential monomorphism, then $\alpha'$ is also an essential monomorphism:

$$
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\alpha'} & B'
\end{array}
$$
Neat Homomorphisms over Dedekind Domains

Theorem

Let \( f : A \to B \) be a homom. of modules. TFAE:

1. \( f : A \to B \) is \( E \)-neat.
2. If

\[
\begin{array}{ccc}
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\]

is a pushout diagram of \( f \) and \( \alpha \), where \( \alpha \) is an essential monomorphism, then \( \alpha' \) is also an essential monomorphism.

Corollary

Let \( K \subseteq A \) be a submodule. Then the epim. \( A \to A/K \) is \( E \)-neat iff \( A/K \owns E(A/K) \).
**Theorem**

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is a pushout diagram of $f$ and $\alpha$, where $\alpha$ is an essential monomorphism, then $\alpha'$ is also an essential monomorphism.

**Corollary**

Let $K \subseteq A$ be a submodule. Then the epim. $A \rightarrow A/K$ is $E$-neat iff $A/K \leq E(A)/K$. 
Z-neat homomorphisms (condition-(2))

We call a homomorphism \( f : A \rightarrow B \) of modules Z-neat if \( \text{Im} \ f \) is closed in \( B \) and \( \text{Ker} \ f \subseteq \text{Rad} \ A \).
Z-neat homomorphisms (condition-(2))

We call a homomorphism \( f : A \to B \) of modules \( \mathbb{Z} \)-neat if \( \text{Im} \ f \) is closed in \( B \) and \( \text{Ker} \ f \subseteq \text{Rad} \ A \).

Proposition [Bowe(1972)]

\( R \) is left hereditary (i.e. every left ideal of \( R \) is projective) iff being the natural decomp. \( f = ip \) with \( p \) epim. and \( i \) mono. \( E \)-neat homomorphism implies that \( p \) and \( i \) are \( E \)-neat homomorphisms.
**Z-neat homomorphisms (condition-(2))**

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$R$ is left hereditary (i.e. every left ideal of $R$ is projective) iff being the natural decomp. $f = ip$ with $p$ epim. and $i$ mono. $E$-neat homomorphism implies that $p$ and $i$ are $E$-neat homomorphisms.

**Proposition**

If $h = gf$ is an $E$-neat homomorphism, then $f$ is always $E$-neat.
Theorem

Over a Dedekind domain, the natural epimorphism $f : A \to A/K$ is $E$-neat iff $K \subseteq \text{Rad } A$. 
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THEOREM: $E$-neat $= Z$-neat over Dedekind domains

Let $R$ be a Dedekind domain. A homomorphism $f : A \to B$ of $R$-modules is $E$-neat if and only if it is $Z$-neat, i.e., $\text{Im} f$ is closed in $B$ and $\text{Ker} f \subseteq \text{Rad} A$. 


THANK YOU FOR YOUR ATTENTION