Neat Homomorphisms over Dedekind Domains

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(Joint work with Engin Mermut)

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Let $\mathcal{P}$ be a class of s.e.s of R-modules.
If $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ belongs to $\mathcal{P}$, then f is called a $\mathcal{P}$-monom., and g is called a $\mathcal{P}$-epim.

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Proper class

$\mathcal{P}$ is said to be proper if

P1. If $\mathbb{E}$ is in $\mathcal{P}$, then $\mathcal{P}$ contains every s.e.s. isomorphic to $\mathbb{E}$.

P2. $\mathcal{P}$ contains all splitting s.e.s.

P3. The composite of two $\mathcal{P}$-monom. (resp. $\mathcal{P}$-epim.) is a $\mathcal{P}$-monom. (resp. $\mathcal{P}$-epim.) if these composites are defined.

P4. If g and f are monom., and gf is a $\mathcal{P}$-monom., then f is a $\mathcal{P}$-monom. Moreover, if g and f are epim. and gf is a $\mathcal{P}$-epim., then g is a $\mathcal{P}$-epim.

Example of proper classes

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Let $\mathcal{M}$ be a class of modules.

proj., inj., flatly generated proper classes

- $\pi^{-1}(\mathcal{M})$: the class of all s.e.s. $\mathbb{E}$ s.t. $\mathrm{Hom}_R(L, \mathbb{E})$ is exact $\forall L \in \mathcal{M}$ (proper class proj. gen. by $\mathcal{M}$);
- $\iota^{-1}(\mathcal{M})$: the class of all s.e.s. $\mathbb{E}$ s.t. $\mathrm{Hom}_R(\mathbb{E}, L)$ is exact $\forall L \in \mathcal{M}$ (proper class inj. gen. by $\mathcal{M}$);
- $\tau^{-1}(\mathcal{M})$: the class of all s.e.s. $\mathbb{E}$ s.t. $L \otimes \mathbb{E}$ is exact $\forall L \in \mathcal{M}$ (proper class flatly gen. by the class $\mathcal{M}$ of right modules).

- A subgroup $A \subseteq B$ is called neat in B if $A \cap pB = pA$ for all primes p [Honda(1956)].
The class of all neat-exact sequences of abelian groups forms a proper class, denoted $\mathcal{N}eat_{\mathbb{Z}}$.

- A subgroup $A \subseteq B$ is called neat in B if $A \cap pB = pA$ for all primes p [Honda(1956)].
The class of all neat-exact sequences of abelian groups forms a proper class, denoted $\mathcal{N}eat_{\mathbb{Z}}$.
- A is closed in B if A has no proper essential extension in B.

$\mathcal{C}losed_{\mathbb{Z}} = \mathcal{N}eat_{\mathbb{Z}}$

The proper class $\mathcal{C}losed_{\mathbb{Z}} = \mathcal{N}eat_{\mathbb{Z}}$ is proj., inj. and flatly generated by all simple groups $\mathbb{Z}/p\mathbb{Z}$ (p prime number):

$$\begin{aligned} \mathcal{C}losed_{\mathbb{Z}} &= \mathcal{N}eat_{\mathbb{Z}} \\ &= \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\ &= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\ &= \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}). \end{aligned}$$

Neat submodules [Stenstrom(1967)]

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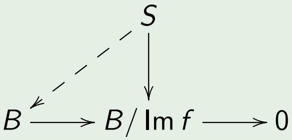

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$$\begin{array}{ccccc} & & S & & \\ & \swarrow & \downarrow & & \\ B & \longrightarrow & B/\operatorname{Im} f & \longrightarrow & 0 \end{array}$$

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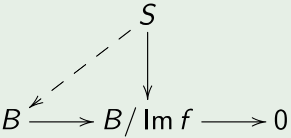

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$\mathcal{N}eat_R$

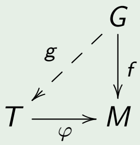
The class of all s.e.s. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ s.t. $\operatorname{Im} f$ is a neat submodule of B forms a proper class:

$\mathcal{N}eat_R = \pi^{-1}(\{\text{ all simple } R\text{-modules}\})$

Torsion free covers of modules [Enochs(1971)]

Over a commutative domain R, a homomorphism $\varphi : T \to M$, where T is a torsion free R-module, is called a torsion free cover of M if

(i) for every torsion free R-module G and a homomorphism $f : G \to M$ there is a homomorphism $g : G \to T$ s.t. $\varphi g = f$:



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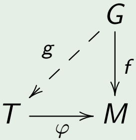

(ii) $\operatorname{Ker} \varphi$ contains *no* non-trivial submodule S of T s.t. $rS = rT \cap S$ for all $r \in R$ (i.e. S is an *RD*-submodule (relatively divisible submodule) of T).

Neat homomorphisms of Enochs and Bowe

A homomorphism $f : M \to N$ is neat if given any proper submodule H of G and any homomorphism $\sigma : H \to M$, the homomorphism $f \circ \sigma$ has a proper extension in G iff σ has a proper extension in G.

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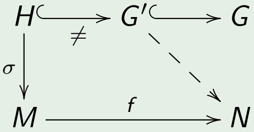
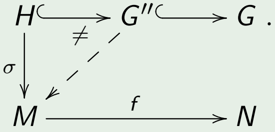


always guarantees the existence of a commutative diagram (2):



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These neat homomorphisms need not be monic or epic. It is different from our definition of neat, so we call them E-neat homomorphisms.

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2. A is a neat subgroup of B iff the monom. $A \hookrightarrow B$ is E-neat.
3. A is a closed submodule of B iff the monom. $A \hookrightarrow B$ is E-neat (i.e. A is E-neat submodule).

C-ring of Renault(1964)

R is called a left C-ring if $\mathrm{Soc}(R/I) \neq 0$, for every essential proper left ideal I of R.

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- R is a left C-ring iff $\mathcal{C}losed = \mathcal{N}eat_R$. [Generolov(1978)]

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- R is a left C-ring iff $\mathcal{C}losed = \mathcal{N}eat_R$. [Generolov(1978)]
- So, over a left C-ring, neat submodules and E-neat submodules coincide.

Theorem [Bowe(1972)]

Let $f : A \to B$ be a homom. of modules. TFAE:

1. f is E-neat.
2. In the defn. of E-neat homom., it suffices to take $G = R$ and H a left ideal of R.
3. In the defn. of E-neat homom., it suffices to take σ a monom. and G as an essential extension of H.
4. There are no proper extensions of f in the injective envelope $E(A)$ of A.

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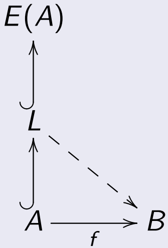

commutates, then $A = L$.

- Since a monomorphism $f : A \to B$ is E-neat iff $\operatorname{Im} f$ is a closed submodule, the class of all s.e.s. $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ s.t. $f : A \to B$ is E-neat forms the proper class $\mathcal{C}losed$.

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- So, we investigate the class of all such s.e.s. s.t. $g : B \to C$ is E-neat, denoted by $\mathcal{EN}eat$.

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Theorem

The projection epimorphism $f : A \oplus B \to A$ is E-neat iff $\operatorname{Ker} f \cong B$ is injective.

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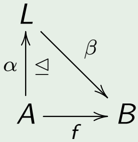
E-neat homomorphisms [Zoschinger(1978)]

A homomorphisms $f : A \to B$ of modules is E-neat if for every decomposition $f = \beta\alpha$ where α is an essential monomorphism, α is an isomorphism:



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Let $f : A \to B$ be a homom. of abelian groups. TFAE:

1. f is E-neat;
2. $\operatorname{Im} f$ is closed in B and $\operatorname{Ker} f \subseteq \operatorname{Rad} A$;
3. $f^{-1}(pB) = pA$ for all prime numbers p;
4. if the following diagram is pushout of f and α, and α is an essential monomorphism, then α' is also an essential monomorphism:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{\alpha'} & B' \end{array}$$

Theorem

Let $f : A \to B$ be a homom. of modules. TFAE:

1. $f : A \to B$ is E-neat.
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Theorem

Let $f : A \to B$ be a homom. of modules. TFAE:

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$$\begin{CD} A @>\alpha>> A' \\ @VfVV @VVf'V \\ B @>>\alpha'> B' \end{CD}$$

is a pushout diagram of f and α, where α is an essential monomorphism, then α' is also an essential monomorphism.

Corollary

Let $K \subseteq A$ be a submodule. Then the epim. $A \to A/K$ is E-neat iff $A/K \trianglelefteq E(A)/K$.

Z-neat homomorphisms (condition-(2))

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Proposition [Bowe(1972)]

R is left hereditary (i.e. every left ideal of R is projective) iff being the natural decomp. $f = ip$ with p epim. and i mono. E-neat homomorphism implies that p and i are E-neat homomorphisms.

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Proposition

If $h = gf$ is an E-neat homomorphism, then f is always E-neat.

Theorem

Over a Dedekind domain, the natural epimorphism $f : A \to A/K$ is E-neat iff $K \subseteq \operatorname{Rad} A$.

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THEOREM: E-neat = Z-neat over Dedekind domains

Let R be a Dedekind domain. A homomorphism $f : A \to B$ of R-modules is E-neat if and only if it is Z-neat, i.e., $\operatorname{Im} f$ is closed in B and $\operatorname{Ker} f \subseteq \operatorname{Rad} A$.

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THANK YOU FOR YOUR ATTENTION