ON DUAL AUTOMORPHISM-INVARIANT MODULES

Serap Şahinkaya
(Joint work with T.C. Quynh)

Gebze Technical University

June 2017
Generalization of Injectivity (Projectivity)
Automorphism Invariant Modules
Dual Automorphism Invariant Modules
Main Results

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Generalized Notions of Injectivity and Projectivity
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   - Dual automorphism $N$-invariant modules
Generalized Notions of Injectivity and Projectivity

Automorphism Invariant Modules

Dual Automorphism Invariant Modules

Main Results

- Dual automorphism $N$-invariant modules
- S-ADS Modules
A module $M$ is called quasi-injective, (or self-injective) if for every submodule $N$ of $M$ every $R$-homomorphism of $N$ into $M$ can be extended to $R$-endomorphism of $M$ (Johnson and Wong, J. London Math. Soc. 1961). It is a generalization of injectivity.

A module $M$ is called quasi-projective if for any epimorphism $g: M \rightarrow M/\mathfrak{T}$ and any morphism $f: M \rightarrow M/\mathfrak{T}$ there exists a homomorphism $h: M \rightarrow M$ such that $f = gh$ (Y. Miyashita, 1966). It is a generalization of projectivity.
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A module $M$ was called $N$-pseudo-injective if for any submodule $A$ of $N$ every monomorphism $f: A \rightarrow M$ can be extended to $g: N \rightarrow M$.

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\[ \begin{array}{ccc}
0 & \rightarrow & A \\
& \downarrow f & \rightarrow \\
& & N \\
& \downarrow g & \rightarrow \\
& & M \\
\end{array} \]

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A module $M$ was called $N$-pseudo-projective if for every submodule $A$ of $M$ and any epimorphism $g: N \rightarrow M/A$ can be lifted to a homomorphism $f: N \rightarrow M$. If $N M M/A \neq 0$ then it is called pseudo-projective (Bican, Acta Mathematica Academiae Scientiarum Hungaricae, 1976).
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$$
\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{g} & & \downarrow{}
\end{array}
$$


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Modules that are invariant under endomorphisms of their injective envelope (generalization of quasi injective modules) were studied by Dickson and Fuller just for the case of finite-dimensional algebras over fields $F$ with more than two elements (Pacific J. Math., 1969).

A module $M$ which is invariant under automorphisms of its injective envelope has been called an automorphism invariant module by Lee and Zhou equivalently $M$ is automorphism-invariant if every isomorphism between two essential submodules of $M$ extends to an automorphism of $M$ (J. Alg. Appl., 2013) (for modules over any ring).

Quasi-injective and pseudo-injective modules modules are automorphism invariant (by Lee and Zhou, J. Algebra Appl., 2013).

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$M$ is automorphism $N$-invariant if for any essential submodule $A$ of $N$, any essential monomorphism $f: A \rightarrow M$ can be extended to some $g \in \text{Hom}(N, M)$ (Quynh and Kosan, J. Alg. App., 2015).

$M$ is called automorphism-invariant if $M$ is automorphism $M$-invariant.

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If $M$ is pseudo-$N$-injective then $M$ is automorphism $N$-invariant.
A right $R$-module $M$ is called dual automorphism-invariant if whenever $K_1$ and $K_2$ are small submodules of $M$, then any epimorphism $\eta: M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism $\phi$ of $M$ (Singh and Srivastava, J. Alg., 2013).

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\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & M \\
\downarrow & & \downarrow \\
M/K_1 & \xrightarrow{\eta} & M/K_2
\end{array}
\]
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Converse is true over right perfect rings (by Guil Asensio, P. A., Keskin Tutuncu, D., Kalebogaz, B., Srivastava, A. K.).
Layout

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Main Results

Dual automorphism $N$-invariant modules
$s$-ADS modules

Rings are associative with unity and modules are unital right $R$-modules.

The purpose of this paper is to initiate the study of dual automorphism $N$-invariant modules via $N$-pseudo-projective modules.

Definition
We call $M$ dual automorphism $N$-invariant if, whenever $K_1$ is a small submodule of $M$ and $K_2$ is a small submodule of $N$, then any epimorphism $p: M/K_1 \to N/K_2$ with small kernel lifts to a homomorphism $\phi: M \to N$. That is:

\[
\begin{array}{c}
\text{M} \\
\downarrow \\
\text{M} \\
\downarrow \\
\text{N} \\
\downarrow \\
\end{array}
\xrightarrow{\phi}
\begin{array}{c}
\text{N} \\
\downarrow \\
\text{N} \\
\downarrow \\
\end{array}
\]

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**Definition**

We call $M$ *dual automorphism $N$-invariant* if, whenever $K_1$ is a small submodule of $M$ and $K_2$ is a small submodule of $N$, then any epimorphism $p : M/K_1 \rightarrow N/K_2$ with small kernel lifts to a homomorphism $\varphi : M \rightarrow N$. That is:

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi} & N \\
\downarrow & & \downarrow \\
M/K_1 & \xrightarrow{\eta} & N/K_2
\end{array}
$$
Theorem (Ş., Quynh)

The following conditions are equivalent for a right $R$-module $M$:

1. $M$ is dual automorphism $N$-invariant.

2. For any small submodule $K_1$ of $M$ and small submodule $K_2$ of $N$, every epimorphism $p : M/K_1 \to N/K_2$ with small kernel lifts to an epimorphism $\phi : M \to N$.

3. For any small submodule $K_2$ of $N$, every epimorphism $f : M \to N/K_2$ with small kernel lifts to a homomorphism $\phi : M \to N$.

\[
\begin{array}{cccc}
M & \overset{\phi}{\to} & N & \overset{f}{\to} \ N/K_2 & \overset{\to}{\longrightarrow} & 0 \\
& & M & \to & N & \to
\end{array}
\]
Corollary (Ş., Quynh)

The following conditions are equivalent for a right $R$-module $M$:

1. $M$ is dual automorphism invariant.

2. For any small submodule $K$ of $M$, every epimorphism $f : M \to M/K$ with small kernel lifts to an endomorphism of $M$.

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & f \\
\downarrow & & \downarrow \\
M & \rightarrow & M/K & \rightarrow & 0
\end{array}
\]
Recall: Let $N$ and $L$ be submodules of $M$. The module $N$ is called a supplement of $L$ in $M$ if $M = N + L$ and $N \cap L \ll N$. $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. 
Recall: Let $N$ and $L$ be submodules of $M$. The module $N$ is called a **supplement** of $L$ in $M$ if $M = N + L$ and $N \cap L \ll N$. $M$ is called **supplemented** if every submodule of $M$ has a supplement in $M$.

**Proposition (Ş., Quynh)**

Let $M$ and $N$ be modules and $X = M \oplus N$. The following conditions are equivalent:

1. $M$ is dual automorphism $N$-invariant.
2. For each submodule $K$ of $X$ such that $N$ is a supplement of $K$ in $X$, there exists $C \leq K$ such that $N \oplus C = X$. 
A module $M$ is called a \textit{hollow} module if every proper submodule of $M$ is small in $M$. The following observation was proved for local modules by Singh and Srivastava (Journal of Algebra, 2012).
A module $M$ is called a *hollow* module if every proper submodule of $M$ is small in $M$. The following observation was proved for local modules by Singh and Srivastava (Journal of Algebra, 2012).

**Proposition (Ş., Quynh)**

Assume that $M_1, M_2$ are two hollow modules. If $M_1$ is dual automorphism $M_2$-invariant, then $M_1$ is $M_2$-projective.
Theorem (Ş., Quynh)

Let $M$ and $N$ be two $R$-modules.

1. Every direct summand of a dual automorphism $M$-invariant module is also dual automorphism $M$-invariant.

2. $M$ is dual automorphism $N$-invariant if and only if any isomorphism $f : M/B \to N/A$ with $B \ll M$ and $A \ll N$ lifts to a homomorphism from $M$ to $N$.

3. If $M$ is a dual automorphism $N$-invariant module and $K \cong N$, then $M$ is dual automorphism $K$-invariant.

4. Assume that $N = A \oplus B$ and $M = C \oplus D$ such that there exists a small epimorphism from $D$ to $B$. If $M$ is dual automorphism $N$-invariant, then $C$ is dual automorphism $A$-invariant.
The following theorem extends Singh and Srivastava (Journal of Algebra, 2012).

**Theorem (S.¸., Quynh)**

Let $\pi_1: P_1 \rightarrow M$ and $\pi_2: P_2 \rightarrow N$ be projective covers. Then the following conditions are equivalent.

1. $M$ is dual automorphism $N$-invariant.
2. $\sigma(Ker(\pi_1)) \leq Ker(\pi_2)$ for any isomorphism $\sigma: P_1 \rightarrow P_2$.

Letting $M = N$, Theorem yields the following corollary:

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Let $\pi: P \rightarrow M$ be a projective cover. Then $M$ is dual automorphism-invariant if and only if $\sigma(Ker(\pi)) \leq Ker(\pi)$ for any isomorphism $\sigma: P \rightarrow P$. 
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By Singh and Srivastava (Journal of Algebra, 2012) any pseudo-projective module is dual automorphism-invariant.
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Any $N$-pseudo-projective module is dual automorphism $N$-invariant.

**Theorem (Ş., Quynh)**

Let $M$ and $N$ be mutually dual automorphism invariant modules and $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to N$ be projective covers. If $P_1 \cong P_2$, then every isomorphism $\sigma : P_1 \to P_2$ reduces an isomorphism from $\text{Ker}(\pi_1)$ to $\text{Ker}(\pi_2)$. 
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   - Dual automorphism $N$-invariant modules
   - $s$-ADS modules
A right module $M$ over a ring $R$ is said to be ADS if for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$, we have $M = S \oplus T'$. (see, Fuchs, Infinite Abelian Groups, 1970)

An $R$-module $M$ is ADS if and only if for each decomposition $M = S \oplus T$, $S$ and $T$ are mutually injective.

A module $M$ is called an $e$-ADS module if, for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$ with $T' \cap T = 0$ and $S \cap (T' \oplus T) \leq eS$, we have $M = S \oplus T'$. (Kosan and Quynh)

$M$ is an $e$-ADS module if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively automorphism invariant.
A right module $M$ over a ring $R$ is said to be ADS if for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$, we have $M = S \oplus T'$. (see, Fuchs, Infinite Abelian Groups, 1970)
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$M$ is an e-ADS module if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively automorphism invariant.
Any module $M$ is called *amply supplemented* if $B$ contains a supplement of $A$ in $M$ whenever $M = A + B$. 
Any module $M$ is called **amply supplemented** if $B$ contains a supplement of $A$ in $M$ whenever $M = A + B$.

**Theorem (Ş., Quynh)**

Assume that an amply supplemented $R$-module $X$ has a decomposition $X = M \oplus N$ for some $R$-modules $M$ and $N$. Then the following conditions are equivalent:

1. $M$ is dual automorphism $N$-invariant.

2. For any supplement $K$ of $N$ in $X$ with $K + M = X$ and $(K \cap M) \ll X$, the module $X$ has a decomposition $X = K \oplus N$.

3. For each submodule $K$ of $X$ such that $K$ is a supplement of $N$ in $X$ and $M$ is a supplement of $K$ in $X$, we have $X = K \oplus N$. 
We call $M$ an \textit{s-ADS-module} if for every decomposition $M = S \oplus T$ of $M$ and every supplement $T'$ of $S$ with $T' + T = M$ and $(T \cap T') \ll M$, we have $M = S \oplus T'$. 
We call $M$ an $s$-ADS-$module$ if for every decomposition $M = S \oplus T$ of $M$ and every supplement $T'$ of $S$ with $T' + T = M$ and $(T \cap T') \ll M$, we have $M = S \oplus T'$.

**Theorem (Ş., Quynh)**

The following conditions are equivalent for a module $M$:

1. $M$ is $s$-ADS.
2. For every decomposition $M = S \oplus T$, if $T'$ is supplement of $S$ in $M$ and $T$ is supplement of $T'$ in $M$, then $M = S \oplus T'$. 
Theorem (Ş., Quynh)

An amply supplemented $R$-module $M$ is s-ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively dual automorphism invariant.
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An amply supplemented $R$-module $M$ is s-ADS if and only if for each decomposition $M = A \oplus B$, $A$ and $B$ are relatively dual automorphism invariant.

Corollary (Ş., Quynh)
Every amply supplemented dual automorphism-invariant module is s-ADS.
A right $R$-module $M$ is said to be ADS if for every decomposition $M = S \oplus T$ and for every supplement $T'$ of $S$, we have $M = S \oplus T'$ (see Keskin, Bull. of Math. Sciences 2012). Clearly every ADS module is s-ADS.

**Theorem (S¸., Quynh)**

The following conditions are equivalent for a ring $R$:

1. $R$ is a right V-ring.
2. Every 2-generated right $R$-module is ADS∗.
3. Every 2-generated right $R$-module is s-ADS.

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A right $R$-module $M$ is said to be $ADS^*$ if for every decomposition $M = S \oplus T$ and for every supplement $T'$ of $S$, we have $M = S \oplus T'$ (see Keskin, Bull. of Math. Sciences 2012). Clearly every $ADS^*$ module is s-ADS.
A right $R$-module $M$ is said to be $ADS^*$ if for every decomposition $M = S \oplus T$ and for every supplement $T'$ of $S$, we have $M = S \oplus T'$ (see Keskin, Bull. of Math. Sciences 2012). Clearly every $ADS^*$ module is s-ADS.

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