On The Semisimplicity And The Submodule Characterizations Of G-set Modules

Mehmet Uc, Mustafa Alkan

Mehmet Akif Ersoy University, Department of Mathematics, Burdur, Turkey
Akdeniz University, Department of Mathematics, Antalya, Turkey

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Introduction

Throughout this talk, G is a finite group with identity element e, R is a commutative ring with unity 1, M is an R–module, RG is the group ring, $H \leq G$ denotes that H is a subgroup of G and S is a G–set with a group action of G on S. If N is an R–submodule of M, it is denoted by $N_R \leq M_R$.

G-set Modules

- MS denote the set of all formal expression of the form $\sum_{s\in S} m_s s$ where $m_s \in M$ and $m_s = 0$ for almost every s.
- For elements $\mu = \sum_{s\in S} m_s s$, $\eta = \sum_{s\in S} n_s s \in MS$, by writing $\mu = \eta$ we mean $m_s = n_s$ for all $s \in S$.
- We define the sum in MS componentwise $\mu + \eta = \sum_{s\in S}(m_s + n_s)s$ and the scalar product of $\sum_{s\in S} m_s s$ by $r \in R$ that is $\sum_{s\in S}(rm_s)s$.

Clearly, MS is an R–module with the sum and the scalar product defined above.

G-set Modules

For $\rho = \sum_{g \in G} r_g g \in RG$, the product of $\sum_{s \in S} m_s s$ by ρ is

$$\begin{aligned} \rho\mu &= \sum_{s \in S} r_g m_s (sg), \ \ sg = s' \in S, \\ &= \sum_{s' \in S} m_{s'} s' \in MS \end{aligned}$$

So, MS is a left module over RG, and also as an R–module, it is denoted by $(MS)_{RG}$ and $(MS)_R$, respectively.

Definition

The RG–module MS is called G–**set module** of S by M over RG.

If S is a G–set and H is a subgroup of G, then S is also an H–set and MS is an RH–module.

If S is a G–set and a group, and $M = R$, then RS is a group algebra. If a group acts on itself by multiplication then naturally we have $(MS)_{RG} = (MG)_{RG}$.

Examples for G-set Modules

Example

Let M be an R–module, $G = D_6 = \langle a, b : a^3 = b^2 = e, b^{-1}ab = a^{-1} \rangle$ and $r = \sum_{g \in D_6} r_g g = r_1 e + r_2 a + r_3 a^2 + r_4 b + r_5 ba + r_6 ba^2 \in RD_6$.

1. Let $S = G$ and let the group act itself by multiplication. Then $MS = MG$ is an RG–module.
2. Let $S = \{K_1 = \{e, b\}, K_2 = \{a, ba\}, K_3 = \{a^2, ba^2\}\}$ that is the set of right cosets of a fixed subgroup $H = C_2 = \langle b : b^2 = e \rangle \leq D_6$ and let G act on S by $g * (Hx) = H(gx)$ for $x, g \in G$. Then $MS = \{\sum_{s \in S} m_s s = m_{K_1} K_1 + m_{K_2} K_2 + m_{K_3} K_3 \mid m_s \in M\}$ and we have the following relations such that

Example Continues

Example

$$
\begin{array}{ccc}
K_1 1 = K_1 & K_2 1 = K_2 & K_3 1 = K_3 \\
K_1 a = K_2 & K_2 a = K_1 & K_3 a = K_1 \\
K_1 a^2 = K_3 & K_2 a^2 = K_3 & K_3 a^2 = K_2 \\
K_1 b = K_1 & K_2 b = K_3 & K_3 b = K_2 \\
K_1 ba = K_2 & K_2 ba = K_1 & K_3 ba = K_3 \\
K_1 ba^2 = K_3 & K_2 ba^2 = K_2 & K_3 ba^2 = K_1.
\end{array}
$$

So, we get

$$
\begin{aligned}
r\mu = & \left(r_1 m_{K_1} + r_4 m_{K_1} + r_3 m_{K_2} + r_5 m_{K_2} + r_2 m_{K_3} + r_6 m_{K_3}\right) K_1 \\
& + \left(r_2 m_{K_1} + r_5 m_{K_1} + r_1 m_{K_2} + r_6 m_{K_2} + r_3 m_{K_3} + r_4 m_{K_3}\right) K_2 \\
& + \left(r_3 m_{K_1} + r_6 m_{K_1} + r_2 m_{K_2} + r_4 m_{K_2} + r_1 m_{K_3} + r_5 m_{K_3}\right) K_3.
\end{aligned}
$$

Some relations on G–set modules

Lemma

Let M be an R–module, N an R–submodule of M, G a finite group, S a G–set. Then $\frac{MS}{NS} \simeq (\frac{M}{N})S$.

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Proof.

We know that NS is an RG–submodule of MS. Define a map θ such that

$$\theta : \quad MS \longrightarrow (\tfrac{M}{N})S \ , \quad \mu = \sum_{s \in S} m_s s \longmapsto \theta(\mu) = \sum_{s \in S} (m_s + N)s$$

$$\begin{aligned} \theta(g\mu) &= \theta(g \sum_{s \in S} m_s s) \\ &= g\theta(\mu) \end{aligned}$$

So, θ is a G–set homomorphism. It is clear that θ is a G–set epimorphism. θ is an RG–epimorphism and we get $\ker \theta = NS$. □

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Let N be an R–submodule of an R–module M, S a G–set. Let I denote the index of disjoint orbits of S, J a subset of I and $S' = \bigcup_{j \in J} Gs_j$ and let Gs_i be an orbit Gs of $s_i \in S$ for $i \in I$. Then we have the following results:

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1. *NGs_i is an RG–submodule of MS for $s_i \in S$. Moreover, NGs_i is a minimal RG–submodule of MS containg N under the action induced from that on S.*

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1. *NGs_i is an RG–submodule of MS for $s_i \in S$. Moreover, NGs_i is a minimal RG–submodule of MS containg N under the action induced from that on S.*
2. $NS' = N(\bigcup_{j \in J} Gs_j) = \bigcup_{j \in J} (NGs_j)$.

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2. $NS' = N(\bigcup_{j \in J} Gs_j) = \bigcup_{j \in J} (NGs_j)$.
3. *NS' is an RG–submodule of MS.*

Proof.

1. It is clear that $NGs_i \subseteq MS$. Let $\eta = \sum_{g\in G} n_g gs_i \in NGs_i$, $r \in R$, $h \in G$. Then we have $r\eta \in NGs_i$ and $h\eta = h(\sum_{g\in G} n_g gs_i) = \sum_{g\in G} n_g hgs_i = \sum_{hg=g'\in G} n_g g' s_i \in NGs_i$. Hence, NGs_i is an RG–submodule of MS. Assume that there is an RG–submodule N_1 of MS such that $N_R \leq (N_1)_{RG} \leq (NGs_i)_{RG}$. Take an element $n \in N$, and so $nhs_i \in N_1$ for some $h \in G$ since $(N_1)_{RG} \leq (NGs_i)_{RG}$. Then $h^{-1}(nhs_i) = (nes_i) = ns_i \in N_1$ and $g(ns_i) = ngs_i \in N_1$ for all $g \in G$. This means that $N_1 = NGs_i$.

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2. 2, 3 are clear by the definition of MS.

□

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1. $L_s = \{x \in M \mid \text{there is } y \in L \text{ such that } y = xs + k,\ k \in MS\}$ *is an R–submodule of M.*

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1. $L_s = \{x \in M \mid \text{there is } y \in L \text{ such that } y = xs + k,\ k \in MS\}$ *is an R–submodule of M.*
2. $S_L = \{s \in S \mid \text{there is } x \in M, \text{ and also } k \in L \text{ such that } y = xs + k \in L\}$ *is a G–set in S under the action induced from that on S.*

Some relations on G–set modules

Lemma

(L2) Let M be an R–module and S a G–set. Let I denote the index of disjoint orbits of S such that $S = \bigcup_{i \in I} Gs_i$ and let Gs_i be an orbit of $s_i \in S$ for $i \in I$. If NGs_i is a simple RG–submodule of MS, then N is a simple R–submodule of M and G is a finite group whose order is invertible in $End_R(M)$ $(|G|^{-1} \in End_R(M))$.

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Lemma

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By these two Lemmas L1 and L2, we get the following theorem.

Theorem

Let L be a simple RG–submodule of MS. Then there is a unique simple R–submodule N of M and a unique orbit Gs such that $L = NGs$.

Proof.

For some $s \in S$, by Lemma L1 L_s is a non-zero R–module. And so, $L_s Gs \neq 0$ is an RG–submodule of L. Since L is simple RG–submodule, we have $L_s Gs = L$. Then, by Lemma L2 L_s is a simple R–submodule of M. Take an element $s' \in S$ such that $L_{s'}$ is non-zero R–submodule of M. Hence, $L_{s'} Gs' = L = L_s Gs$. Take an element $x \in L_{s'} Gs'$. And so, we write

$$x = \sum_{i=1}^{n} l_i g_i s' = \sum_{i=1}^{n} k_i g_i s$$

where $l_i \in L_{s'}$, $k_i \in L_s$, $g_i \in G$ and $n = |G|$. Then, there exists $g_j \in G$ such that $g_1 s = g_j s'$, and $s = g_1^{-1} g_j s'$. So, we get $Gs = Gs'$. That is why we can write

$Gs = S_L = \{s \in S \mid \text{there is } x \in M, \text{ and also } k \in L \text{ such that } y = xs + k \in$

Moreover, $N = L_s = L_{s'}$ is unique by the definition of MS. □

Some relations on G–set modules

On the other hand, the following example shows that the converse of the theorem above does not hold.

Example

Let $R = \mathbb{Z}_3$, $M = \mathbb{Z}_3$, $G = C_2 = \langle a : a^2 = e \rangle$ and $RG = \mathbb{Z}_3 C_2$. If $S = G$ and G acts on itself by group multiplication then $MS = \mathbb{Z}_3 C_2$ where $\mathbb{Z}_3 C_2$ is semisimple RG–module since $|G| \leq \infty$ and characteristic of R does not divide $|G|$ by Maschke's Theorem. Since $\mathbb{Z}_3 C_2$ is semisimple there is a unique decomposition of $\mathbb{Z}_3 C_2$ by Artin-Weddernburn Theorem. Then, $\mathbb{Z}_3 C_2 \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3$ as R–module since $|C_2| = 2$. Here, $\mathbb{Z}_3$ is a simple R–submodule of $\mathbb{Z}_3 C_2$. We have $\mathbb{Z}_3 C_2 \simeq \mathbb{Z}_3 C_2(\frac{1+a}{2}) \oplus \mathbb{Z}_3 C_2(\frac{1-a}{2})$ as RG–module where $\mathbb{Z}_3 C_2(\frac{1+a}{2})$ and $\mathbb{Z}_3 C_2(\frac{1-a}{2})$ are simple RG–submodules of $\mathbb{Z}_3 C_2$. Let $N = \mathbb{Z}_3$ that is a simple R–submodule of M. Hovewer, $NGs = \mathbb{Z}_3 C_2$ is not simple RG–module.

Decomposition of G-set modules

Let $e_H = \frac{\hat{H}}{|H|}$, where $|H|$ is the order of H and $\hat{H} = \sum_{h \in H} h$. $End_{RG}MS$ denotes all the RG–endomorphisms of MS.

Lemma

(L3) Let M be an R-module and H a normal subgroup of finite group G. If $|H|$, the order of H, is invertible in R then $\widetilde{e}_H = \frac{\hat{H}}{|H|}$ is an idempotent in $End_{RG}(MS)$. Moreover, $\widetilde{e}_H$ is central in $End_{RG}(MS)$.

Decomposition of G-set modules

Let H be a normal subgroup of G. It is well known that on G/H we have the group action $g(tH) = gtH$ for $g, t \in G$. Consider $g(\sum_{s \in S} m_s(sH)) = (\sum_{s \in S} m_s(gsH))$ for $m_s \in M$.

Let $S' \subset S$ be a G/H–set. Then $S' = \bigcup_{j \in J} G/Hs'_j$ where J denotes the index of disjoint orbits of S' and $MS' = M(\bigcup_{j \in J} G/Hs'_j)$. Then for $\eta = \sum_{s' \in S'} m_{s'} s' \in MS$, we can write $\eta = \sum_{j \in J} \sum_{s' \in G/Hs'_j} m_{s'} s'$. Hence,

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Lemma

Let M be an R–module, G a finite group, H a normal subgroup of G, S a G–set and $S' \subset S$ a G/H–set. Then MS' is an RG–module with action defined as

$$g\eta = g(\sum_{j\in J} \sum_{s'\in G/Hs'_j} m_{s'}s') = \sum_{j\in J} \sum_{s'\in G/Hs'_j} m_{s'}(gtHs'_j)$$

Decomposition of G-set modules

Theorem

Let H be a normal subgroup of G, $|H|$ invertible in R and $\widetilde{e}_H$, defined above, then we have $MS = \widetilde{e}_H.MS \oplus (1 - \widetilde{e}_H).MS$ and there exists a G/H–set $S' \subset S$ such that $\widetilde{e}_H.MS \simeq MS'$. More precisely,

$$\widetilde{e}_H.MS = \widetilde{e}_H \left(M(\bigcup_{i \in I} Gs_i) \right) \simeq M(\bigcup_{i \in I} \widetilde{e}_H Gs_i)$$

Decomposition of G-set modules

Proof.

Firstly, we know that $MG = \widetilde{e}_H.MG \oplus (1 - \widetilde{e}_H).MG$ and $\widetilde{e}_H.MG \simeq M(G/H)$ by the theorem in $[Uc, Alkan, 2017]$. Since $\widetilde{e}_H$ is a central idempotent by Lemma L3, we get $MS = \widetilde{e}_H.MS \oplus (1 - \widetilde{e}_H).MS$. Now, consider $\theta : G \longrightarrow G.\widetilde{e}_H$ where $g \mapsto g\widetilde{e}_H$. This is a group homomorphism since $\theta(gh) = gh\widetilde{e}_H = gh\widetilde{e}_H^2 = g\widetilde{e}_H h\widetilde{e}_H = \theta(g)\theta(h)$. It is clear that θ is a group epimorphism. We have $ker\theta = \{g \in G \mid g\widetilde{e}_H = \widetilde{e}_H\} = \{g \in G \mid (g-1)\widetilde{e}_H = 0\} = H$ since $(g-1)\frac{\hat{H}}{|H|} = 0$ and $g\hat{H} = \hat{H}$ for $g \in H$. Moreover, we get $\frac{G}{er\theta} = \frac{G}{H} \simeq \mathrm{Im}\theta$ $= G\widetilde{e}_H$. So,

$$\widetilde{e}_H.MS = \widetilde{e}_H \left(M(\bigcup_{i \in I} Gs_i) \right) = M(\bigcup_{i \in I} G\widetilde{e}_H s_i) \simeq M(\bigcup_{i \in I} (G/H)s_i)$$

□

Proof Continues

Proof.

Since $gHs_i = gHs_l$ for $s_i, s_l \in S$, $i, l \in I$, we get a G/H–set $S' \subset S$ where $\bigcup_{j \in J} (G/H)s_j = S' \subseteq S$. Hence

$$\widetilde{e}_H.MS \simeq M(\bigcup_{i \in I}(G/H)s_i) = M(\bigcup_{j \in J}(G/H)s_j) = MS'$$

So, $\widetilde{e}_H.MS \simeq MS'$. □

In the theory of the group ring, the augmentation ideal denoted by $\triangle(RG)$ is the kernel of the usual augmentation map ε_R such that

$$\varepsilon_R : \quad RG \longrightarrow R \;, \; \sum_{g \in G} r_g g \longmapsto \sum_{g \in G} r_g \;.$$

The augmentation ideal is always the nontrivial two-sided ideal of the group ring and we have $\triangle(RG) = \left\{ \sum_{g \in G} r_g(g-1) : r_g \in R, g \in G \right\}$.

The augmentation ideal $\triangle(RG)$ is of use for studying not only the relationship between the subgroups of G and the ideals of RG but also the decomposition of RG as direct sum of subrings.

Augmentation Map on MS

Definition

The map

$$\varepsilon_{MS} : \quad MS \longrightarrow M \;, \quad \sum_{s \in S} m_s s \longmapsto \sum_{s \in S} m_s$$

is called **augmentation map on** MS. The kernel of ε_{MS} is denoted by $\triangle_G(MS)$.

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Lemma

Let M be an R–module, G a group and S a G–set. Then $\varepsilon_{MS}(r\mu) = \varepsilon(r)\varepsilon_{MS}(\mu)$ for $\mu = \sum_{s \in S} m_s s \in MS$, $r = \sum_{g \in G} r_g g \in RG$. In particular, ε_{MS} is an R–homomorphism.

Definition

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Theorem

Let M be an R–module, H a subgroup of G, $|H|$ invertible in R, S a G–set and $\widetilde{e}_H$, defined above. Then, $\triangle_{G,H}(MS)$ is an RG–module and $\triangle_{G,H}(MS) = (1 - \widetilde{e}_H).MS$.

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Furthermore, we write $\triangle_{G,G}(MS) = \triangle_G(MS)$. $ker(\varepsilon_{MS}) = \triangle_G(MS)$ and we have $ker(\varepsilon_{MS}) = \triangle_G(MS) = (1-\widetilde{e}_G).MS$.
We know that $\triangle_R(G)$ is the augmetation ideal of RG and for a normal subgroup N of G, $\triangle_R(G,N)$ denote the kernel of the natural epimorphism $RG \longrightarrow R(G/N)$ induced by $G \longrightarrow G/N$. Moreover, $\triangle_R(G,N)$ is a two-sided ideal of RG generated by $\triangle_R(N)$.

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Proof.

We know that $\triangle_R(N) = \{ \sum_{n \in N} r_n(n-1) \mid r_n \in R\}$ and $\triangle_{G,H}(MS) = \{ \sum_{h \in H} (h-1)\mu_h \mid \mu_h \in MS\}$. For $\alpha = \sum_{n \in N} r_n(n-1) \in \triangle_R(N)$, $\mu = \sum_{s \in S} m_s s \in MS$,

$$\begin{aligned} \alpha\mu &= \left(\sum_{n \in N} r_n(n-1)\right)\left(\sum_{s \in S} m_s s\right) = \sum_{n \in N}(n-1)\left(\sum_{s \in S}(r_n m_s)s\right) \\ &= \sum_{n \in N}(n-1)\mu_n \end{aligned}$$

where $\mu_n = \sum_{s \in S}(r_n m_s)s \in MS$. □

Semisimple G–set Modules

Lemma

(L4) Let M be a nonzero R–module, G a group, S a G–set. If $X \cap \triangle_G(MS) = 0$ for some nonzero RG–submodule X of $(MS)_{RG}$, then G is a finite group.

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Proof.

Firstly, we know that $\triangle_G(MS)$ is an RG–submodule of $(MS)_{RG}$. Assume that G is an infinite group. Then for any $0 \neq x = m_1 s_1 + ... + m_k s_k \in X$ where $s_1, ..., s_k \in S$ are distinct and $m_i s_i \neq 0$, there is an element g of G such that $s_i g \neq s_j$ for $1 \leq i \leq k$. Hence,
$(1-g)x = \sum_{s_i \in S} m_i s_i - \sum_{s_i \in S} m_i g s_i \neq 0$, and also $(1-g)x \in Y$. On the other hand,
$0 \neq (1-g)x = \sum_{s_i \in S} m_i (s_i - 1) - \sum_{s_i \in S} m_i (g s_i - 1) \in \triangle_G(MS)$. Then, $X \cap \triangle_G(MS) \neq 0$ and this is a contradiction.

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$(1-g)x = \sum\limits_{s_i \in S} m_i s_i - \sum\limits_{s_i \in S} m_i g s_i \neq 0$, and also $(1-g)x \in Y$. On the other hand,
$0 \neq (1-g)x = \sum\limits_{s_i \in S} m_i(s_i - 1) - \sum\limits_{s_i \in S} m_i(g s_i - 1) \in \triangle_G(MS)$. Then,
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Let M be a nonzero R–module, G a group, S a G–set. Then, MS is a semisimple module over RG if and only if M is a semisimple R–module, G is a finite group whose order is invertible in $End_R(M)$ ($|G|^{-1} \in End_R(M)$).

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Let M be a nonzero R–module, G a group, S a G–set. Then, MS is a semisimple module over RG if and only if M is a semisimple R–module, G is a finite group whose order is invertible in $End_R(M)$ ($|G|^{-1} \in End_R(M)$).

Proof.

Assume that M is a semisimple R–module, G is a finite group whose order is invertible in $End_R(M)$. Let Y be an RG–submodule of MS. Firstly, $(MS)_R$ is semisimple since M_R is semisimple. Hence, Y_R is a direct summand of $(MS)_R$. Moreover, $|G|^{-1} \in End_R(MS)$ since G is finite and $|G|^{-1} \in End_R(M)$. So, Y_{RG} is a direct summand of $(MS)_{RG}$ by Lemma L5 (Lam, 2001) that means $(MS)_{RG}$ is semisimple. □

Proof Continuous

Proof.

Assume that MS is a semisimple module over RG. $\triangle_G(MS)$ is an RG–submodule of MS and we know that $\triangle_G(MS) \neq MS$. So, $\triangle_G(MS)$ is a proper direct summand of $(MS)_{RG}$. Hence, G is a finite group by Lemma L4.

Let N be an R–submodule of M. Then, $(NS)_{RG}$ is an RG–submodule of $(MS)_{RG}$. $(NS)_{RG}$ is a direct summand of $(MS)_{RG}$ because $(MS)_{RG}$ is semisimple, so there is $\alpha^2 = \alpha \in End_{RG}(MS)$ such that $NS = \alpha(MS)$. Let $\alpha\mid_M$ be the restriction of α. Consider the composition such that $\gamma : M \xrightarrow{\alpha|_M} MS \xrightarrow{\varepsilon_{MS}} M$, and so $\gamma \in End_R(M)$. It is clear that $\gamma(M) \subseteq N$. For any $z \in N$, write $z = \alpha(y)$ where $y \in MG$. Then $\gamma(z) = \varepsilon_{MS}\alpha(\alpha(y)) = \varepsilon_{MS}\alpha(y) = \varepsilon_{MS}(z) = z$. Hence, $N = \gamma(M)$, $\gamma(\gamma(z)) = \gamma(z) = z$ and $\gamma^2 = \gamma$ which means that N is a direct summand of M. Therefore, M_R is semisimple R–module. □

Proof Continuous

Proof.

Assume that $|G|^{-1} \notin End_R(M)$. Then there is a prime divisor p of $|G|$ such that $p^{-1} \notin End_R(M)$. We prove that $p : M \longrightarrow M$ is not one-to-one. Indeed, if $p : M \longrightarrow M$ is one-to-one, then $pM \neq M$ because $p^{-1} \notin End_R(M)$. $M = pM \oplus Z$ for some nonzero R–submodule Z of M because M_R is semisimple. Since $pM \cap Z = 0$, we get $pZ = 0$. Thus, $p : M \longrightarrow M$ is not one-to-one. So, there exists a nonzero direct summand N of M_R such that $pN = 0$ because M_R is semisimple. □

Proof Continuous

Proof.

Now consider $N\hat{G}$ that is an RG–submodule of $(MS)_{RG}$ and $N\hat{G} \subseteq \triangle_G(NS)$ since $|G|\, N = 0$. We claim that $\triangle_G(NS)$ is an essential RG–submodule of $(NS)_{RG}$. Let $\sum\limits_{s\in S} n_s s \in NS \backslash \triangle_G(NS)$. Then, $0 \neq \sum\limits_{s\in S} n_s \in N$, and thus $(\sum\limits_{s\in S} n_s s)\hat{G} = (\sum\limits_{s\in S} n_s)\hat{G}$ is a nonzero element of $\triangle_G(NS)$. So $\triangle_G(NS)$ is an essential RG–submodule of $(NS)_{RG}$. Since MS is a semisimple module over RG by hypothesis and $(NS)_{RG}$ is submodule of $(MS)_{RG}$, $(NS)_{RG}$ is semisimple RG–module. Hence, $NS = \triangle_G(NS)$, and so $0 = \varepsilon_{MS}(\triangle_G(NS)) = \varepsilon_{MS}(NS) = N$. This is a contradiction. So, $|G|^{-1} \in End_R(MS)$. □

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THANK YOU.