The Gruenberg-Kegel graph of finite solvable cut groups

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Non-Commutative Rings and Applications VII

1Joint with A. Bächle, A. Kiefer and S. Maheshwary. Partially supported by the Spanish Government under Grant MTM2016-77445-P with ”Fondos FEDER” and, by Fundación Séneca of Murcia under Grant 19880/GERM/15.
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$RG = \bigoplus_{g \in G} Rg$, Group ring of $G$ with coefficients in $R$. 

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Object of study: Group of units of $\mathbb{Z}^G$. 

Trivial units: $\pm G$. 
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Definition

$G$ is cut if every central unit of $\mathbb{Z}G$ is trivial.

Theorem

The following conditions are equivalent:

1. $G$ is cut.
2. If $g \in G$ then every generator of $\langle g \rangle$ is conjugate to $g$ or $g^{-1}$ [Ritter-Sehgal, 2005].
3. The center of every simple epimorphic image of $QG$ is contained in an imaginary quadratic field [Ferraz, 2004].
4. For every $\chi \in \text{Irr}(G)$, the field $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G)$ is contained in an imaginary quadratic field [Bächle-Caicedo-Jespers-Maheshwary, 2021].
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\( G \) is \textit{rational} if the entries of the character table of \( G \) are rational.

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While only \( 0.57\% \) of all groups up to order 512 are rational, \( 86.62\% \) are cut.
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The Gruenberg-Kegel graph

Gruenberg-Kegel graph = GK-graph = Prime graph:
$G$ non-necessarily finite group.

$\Gamma_{GK}(G) : \begin{cases} 
\text{Vertices: } \pi(G) = \{|g| : g \in G, |g| \text{ prime}\}; \\
\text{Edges: } p - q \text{ with } p \neq q, pq = |g| \text{ for some } g \in G.
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\[ \Gamma_{\text{GK}}(G) = \Gamma_{\text{GK}}(V(\mathcal{U}G))? \]

Theorem (Kimmerle, 2006) (PQ) holds for solvable groups.

Theorem (Kimmerle-Konovalov, 2015) (PQ) holds for \( G \) if and only if it holds for every almost simple epimorphic image of \( G \).

(PQ) has been proved for many almost simple groups including symmetric and alternating groups and several sporadic simple groups [Bächle, Margolis, Konovalov, Bovdi, ...].
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Known facts
- If $G$ is a rational and solvable then $\pi(G) \subseteq \{2, 3, 5\}$ [Gow, 1976].
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- Study (PQ) for cut groups and rational groups.

Known facts

- If $G$ is a rational and solvable then $\pi(G) \subseteq \{2, 3, 5\}$ [Gow, 1976].
- If $G$ is a cut and solvable then $\pi(G) \subseteq \{2, 3, 5, 7\}$ [Bachle, 2018].
GK-graphs of finite solvable cut groups: At most 3 vertices

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

GK-graphs of non-trivial solvable cut groups with at most 3 vertices:

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>2 ●</th>
<th>(B)</th>
<th>● 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td>2 ● ● ● 3</td>
<td>(D)</td>
<td>2 ● ●● 3</td>
<td></td>
</tr>
<tr>
<td>(E)</td>
<td>2 ● 5</td>
<td>(F)</td>
<td>2 ● 5</td>
<td></td>
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<tr>
<td>(G)</td>
<td>2 ● 7</td>
<td>(H)</td>
<td>2 ● 3</td>
<td></td>
</tr>
<tr>
<td>(I)</td>
<td>2 ● 5</td>
<td>(J)</td>
<td>2 ● 3</td>
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</tr>
<tr>
<td>(K)</td>
<td>2 ● 3</td>
<td>(L)</td>
<td>2 ● 7</td>
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<tr>
<td>(M)</td>
<td>2 ● 7</td>
<td>(N)</td>
<td>2 ● 7</td>
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<tr>
<td>(O)</td>
<td>2 ● 7</td>
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Possible GK-graphs of finite solvable cut groups with more than 3 vertices.

<table>
<thead>
<tr>
<th>Verified</th>
<th>(P)</th>
<th>2 - - 3</th>
<th>(Q)</th>
<th>2 - - 3</th>
<th>(R)</th>
<th>2 - - 3</th>
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<tbody>
<tr>
<td></td>
<td>5 - - 7</td>
<td></td>
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<tr>
<td>Possible</td>
<td>(S)</td>
<td>2 - - 3</td>
<td>(T)</td>
<td>2 - - 3</td>
<td>(V)</td>
<td>2 - - 3</td>
</tr>
<tr>
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**Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)**

*Possible GK-graphs of non-trivial solvable rational groups:*

<table>
<thead>
<tr>
<th>Verified</th>
<th>Possible</th>
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<tbody>
<tr>
<td>(A) $\mathbb{Z}_2$</td>
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</tr>
<tr>
<td>(C) $\mathbb{Z}_2 \times \mathbb{Z}_3$</td>
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</tr>
<tr>
<td>(D) $\mathbb{Z}_2 \times \mathbb{Z}_3$</td>
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Application 1: GK-graphs of supersolvable rational groups

Theorem (Bächle-Kiefer-Maheshwary-dR, 2021)

The following are equivalent for a graph $\Gamma$.

1. $\Gamma = \Gamma_{GK}(G)$ for some non-trivial metacyclic rational group $G$.
2. $\Gamma = \Gamma_{GK}(G)$ for some non-trivial metabelian rational group $G$.
3. $\Gamma = \Gamma_{GK}(G)$ for some non-trivial supersolvable rational group $G$.
4. $\Gamma = \Gamma_{GK}(G)$ for some non-trivial nilpotent-by-abelian rational group $G$.
5. $\Gamma$ is one of the graphs (A), (C) or (D).
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(PQ) holds for cut groups without an epimorphism image isomorphic to the monster group.
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Corollary

(PQ) holds for rational groups.
Thanks for your attention!
Merci pour votre attention!
Ilginiz için teşekkürler
¡Gracias por su atención!