ENDOMORPHISM RINGS VIA MINIMAL MORPHISMS

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Endomorphism rings via minimal morphisms. Mediterranean J. Math

1. PRELIMINARIES

- R ring with unit.
- Mod-R: category of right modules.
- $J(M)$ the Jacobson radical of the module M.

> PROBLEM
>
> Study the endomorphism ring of the modules belonging to certain class $\mathcal{X}$.

1.1. One classical idea

- Take $M \in \mathcal{X}$.
- Take the injective hull $u : M \to E(M)$.
- Transfer properties from $End_R(E(M))$ to $End_R(M)$.

1.2. One recent application: right continuous modules

Exchange rings

For each $a \in R$, there exists $e^2 = e \in R$ with $Re \leq Ra$ and $R(1-e) \leq R(1-a)$ (Notice $Ra + R(1-a) = R$).

Left strongly exchange rings

For each $a_i, b_i \in R$ with $Ra_i + Rb_i = R$, there exists $e = e^2 \in R$ with

$$Ra_1 \geq Ra_2 \geq Ra_3 \geq \cdots \geq Re \quad \text{and} \quad Rb_1 \geq Rb_2 \geq Rb_3 \geq \cdots \geq R(1-e)$$

Endomorphism ring of continuous modules

- If E is injective, then $End_R(E)$ is left strongly exchange.
- **[Cortés-Izurdiaga, Guil-Asensio]** If M is continuous, then $End_R(M)$ is left strongly exchange.

1.3. One classical aplication: Quasi-injective modules

- If E is injective, $End_R(E)$ is
  - *Semiregular*: regular modulo the Jacobson radical J with lifting idempotents.
  - *Right self-injective* modulo the Jacobson radical.
- **[Faith, Utumi].** If M is quasi-injective, $End_R(M)$ enjoys these properties.

> KEY RESULT: M is quasi-injective if and only if M is a fully invariant submodule of $E(M)$.

Fully invariant submodule

$K \leq M$ satisfies $f(K) \leq K$ for each $f \in End_R(M)$.

1.4. Two recent extensions: automorphism invariant submodules

- **[Guil-Asensio, Srivastava]** If M is an automorphism-invariant submodule of $E(M)$ then $End_R(M)$ is:
  - *Semiregular*: regular modulo the Jacobson radical J with lifting idempotents.
- **[Guil-Asensio, Keskin, Srivastava]** If M has a $\mathcal{X}$-envelope $M \to X$, M is automorphism invariant and $End_R(X)$ is semiregular, then $End_R(M)$ is
  - *Semiregular*.

Automorphism-invariant submodule

$K \leq M$ satisfies $f(K) \leq K$ for each $f \in End_R(M)$ automorphism.

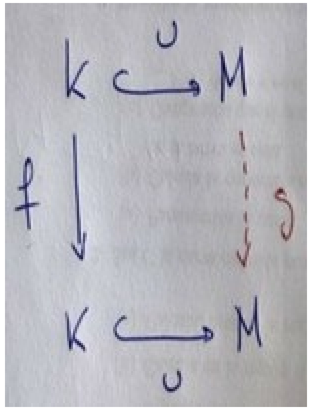
2. OUR WORK

> If we have an inclusion $K \hookrightarrow M$, which is the relationship between $End_R(M)$ and $End_R(K)$?

2.1. Subrings of endomorphisms associated to an inclusion

Given an inclusion $u : K \hookrightarrow M$:

1. $End_R^M(K) =$ Endomorphisms of K that extend to M.



2. $End_R^K(M) =$ Endomorphisms of M which are extensions of endomorphisms of K.

$$End_R^K(M) = f \text{ with } f(K) \leq K$$

3. $\overline{End}_R^K(M) =$ Endomorphisms of M which vainsh at K.

2.2. An easy observation

There is an isomorphism

$$\Phi : End_R^M(K) \rightarrow \frac{End_R^K(M)}{\overline{End}_R^K(M)}$$

defined:

- Take $f \in End_R^M(K)$
- Take an extension g of f to M.
- Define $\Phi(f) = g + \overline{End}_R^K(M)$.

2.3. The radicals come into the scene

- If u is left minimal, then $\overline{End}_R^K(M) \leq J(End_R^K(M))$.
- We get an epimorphism

$$\Gamma : End_R^M(K) \to \frac{End_R^K(M)}{\overline{End}_R^K(M)} \to \frac{End_R^K(M)}{J(End_R^K(M))}$$

with $Ker\Gamma = J(End_R^M(K))$.

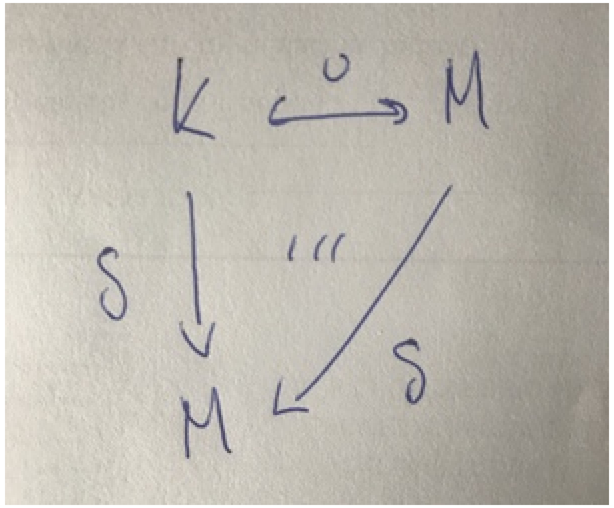
2.4. Main result

If $u : K \to M$ is left minimal then

$$\frac{End_R^M(K)}{J(End_R^M(K))} \cong \frac{End_R^K(M)}{J(End_R^K(M))}$$

Left minimal morphisms

$u : K \to M$ is left minimal if any $g : M \to M$ such that $gu = u$ is an isomorphism.

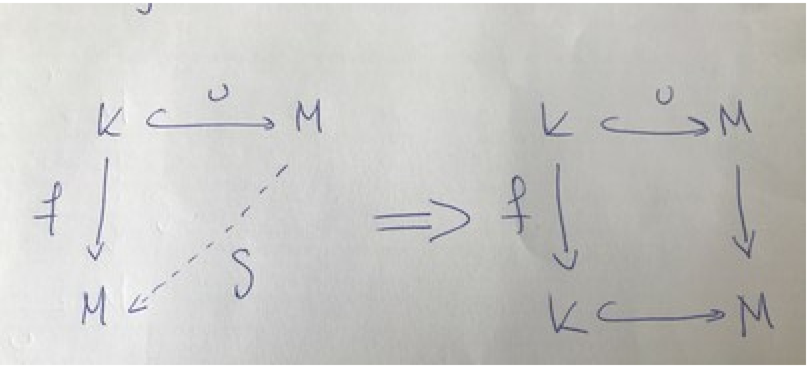


3. APPLICATIONS

3.1. Comparing with the "complete" endomorphisms rings

Let $u : K \to M$ be minimal.

1. If u is M-injective, then $End_R^M(K) = End_R(K)$.



2. If K is fully invariant, then $End_R^K(M) = End_R(M)$.

In this case

$$\frac{End_R(K)}{J(End_R(K))} \cong \frac{End_R(M)}{J(End_R(M))}$$

3.2. The case of envelopes

[Guil-Asensio, Keskin, Srivastava] If $u : K \to X$ is a monic $\mathcal{X}$-envelope and K is fully invariant in X, then

$$\frac{End_R(K)}{J(End_R(K))} \cong \frac{End_R(X)}{J(End_R(X))}$$

$\mathcal{X}$-preenvelopes

$u : K \to X$ is $\mathcal{X}$-envelope if:

1. $X \in \mathcal{X}$.
2. u is X'-injective for each $X' \in \mathcal{X}$.
3. u is left minimal.

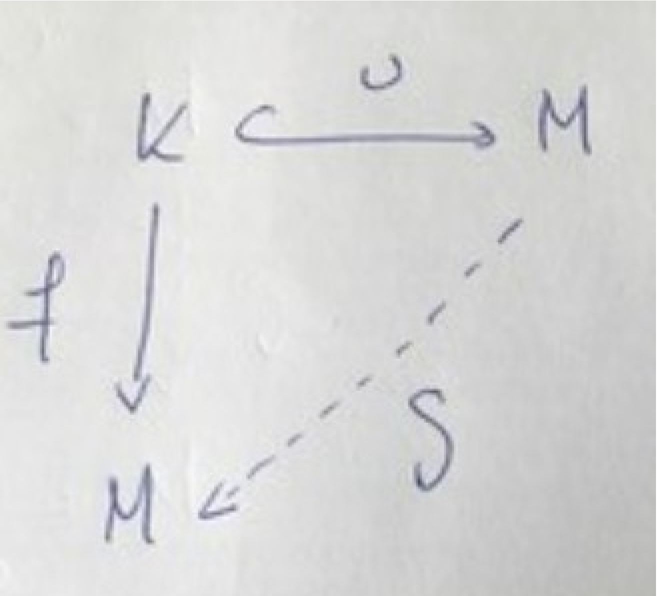
3.3. Fully invariant submodules of $\mathcal{X}$-envelopes

> Which is the class $\mathcal{F}_{\mathcal{X}}$ consisting of modules M which are fully invariant in its $\mathcal{X}$-envelope?

1. $\mathcal{X}$ = Injectives $\Rightarrow \mathcal{F}_{\mathcal{X}}$ = Quasi-injectives.
2. $\mathcal{X}$ = Pure-injectives $\Rightarrow \mathcal{F}_{\mathcal{X}} \subseteq$ Quasi-pure-injectives.

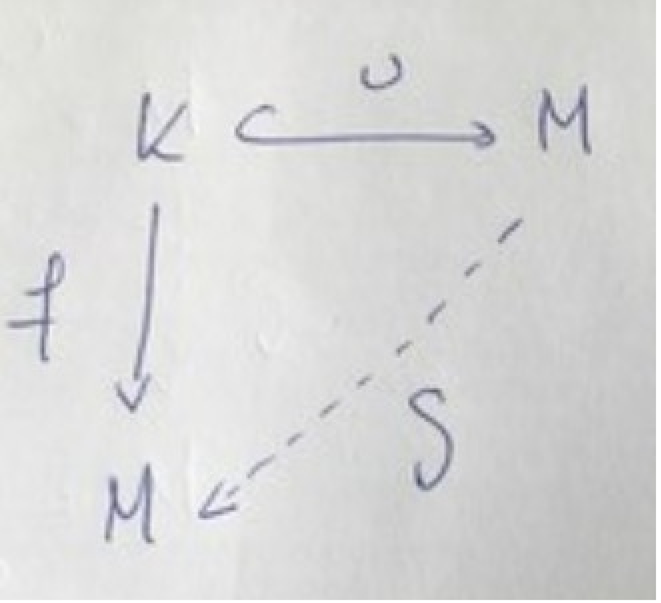
Quasi-pure-injective

M quasi-pure injective if for any pure mono $u : K \to M$ and $f : K \to M$:



Quasi-pure-injective

M quasi-pure injective if for any **pure mono** $u : K \to M$ and $f : K \to M$:

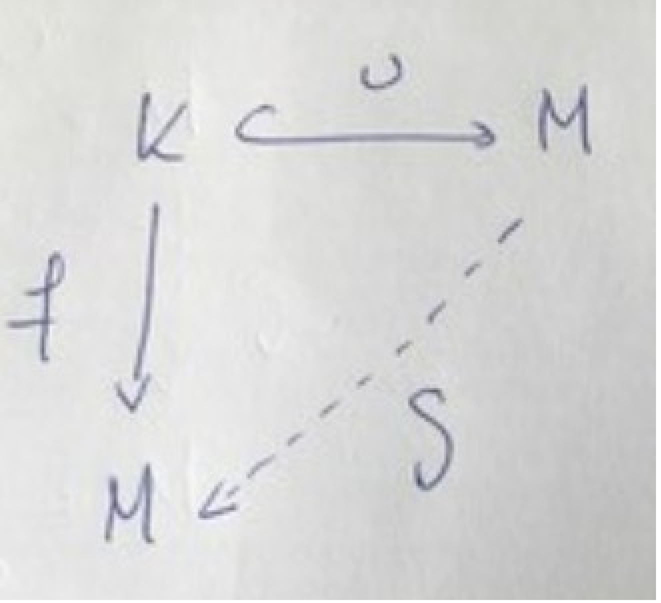


> OPEN PROBLEM (FUCHS): Characterize quasi-pure injective abelian groups.

3. If $\mathcal{X}$ = Cotorsion modules $\Rightarrow \mathcal{F}_{\mathcal{X}} \subseteq$ Quasi-weakly-pure-injectives.

Quasi-weakly-pure-injective

M quasi-weakly-pure injective if for any **strongly pure mono** $u : K \to M$ and $f : K \to M$:



> Strongly-pure monos $\subseteq$ Pure monos

3.4. Cyclic ideals in commutative rings

Main result

If $u : K \to M$ is left minimal then

$$\frac{End_R^M(K)}{J(End_R^M(K))} \cong \frac{End_R^K(M)}{J(End_R^K(M))}$$

> When can we apply it to $I \leq R$.

1. If R is commutative, I is fully invariant $\Rightarrow End_R^I(R) = End_R(R) = R$.
2. If I is cyclic, $End_R^R(I) = End_R(I)$.
3. If $R/I = S_1 \oplus \cdots \oplus S_n$ with S_i non-projective and simple $\Rightarrow I \hookrightarrow R$ is minimal.

$$\frac{End_R(I)}{J(End_R(I))} \cong \frac{R}{J(R)}$$

3.5. Commutative local rings with cyclic radical

If R is a commutative local ring which is not a field and $J(R)$ is cyclic then:

$$\frac{End_R(J(R))}{J(End_R(J(R)))} \cong \frac{R}{J(R)}$$

4. AUTOMORPHISM INVARIANT SUBMODULES

> What happens with a minimal monomorphism $u : K \to M$ with K automorphism invariant in M?

Automorphism invariant

$f(K) \leq K$ for each automorphism $f : M \to M$.

Main result

If $u : K \to M$ is left minimal then

$$\frac{End_R^M(K)}{J(End_R^M(K))} \cong \frac{End_R^K(M)}{J(End_R^K(M))}$$

1. If u is M-injective, then $End_R^M(K) = End_R(K)$.
2. We do not have $End_R^K(M) = End_R(M)$!

4.1. Automorphism invariant submodules

If K is automorphism invariant in M:

1. Idempotents lift modulo J in $End_R(M) \Rightarrow$ so do in $End_R(K)$.
2. $End_R(M)$ semiregular and self-injective modulo the radical $\Rightarrow End_R(K)$ semiregular.

Thank you very much for your attention!