A generalization of Projective Covers

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Join work with W.K.NICHOLSON and A.Ç.ÖZCAN

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– For an ideal I, R is I–semiregular if and only if R/I is regular and I is strongly lifting.

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Now we extend the notion of δ–small submodules and $\delta(M)$ to study a generalization of semiregular rings.

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- Any summand of M is DM in M.

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But the converse is not true in general.

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- A module has a projective 0–cover if and only if M is projective.

Theorem

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Hence S is semisimple and so $Kerf \ll_\delta P$.

This theorem shows that a projective I–cover is a generalization of a projective cover, This theorem shows that a projective I–cover is a generalization of a projective cover, a projective Soc–cover, This theorem shows that a projective I–cover is a generalization of a projective cover, a projective Soc–cover, and projective δ–cover.

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Also we extend some well–known theorems about projective modules.

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$ii)$ $M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.

Then $i) \Rightarrow ii)$,

Lemma

Let M be a projective module and $N \leq M$. Consider the following conditions:

$i)$ M/N has a projective I–cover,

$ii)$ $M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.

Then $i) \Rightarrow ii)$,
if M is DM for I, then $ii) \Rightarrow i)$.

Proof

$(i) \Rightarrow (ii)$

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$$M/N$$

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$$f : Q \longrightarrow M/N \longrightarrow 0$$

such that $Ker\ f \subseteq IQ$ and $Kerf$ is DM in Q.

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There is a decomposition $M/Kerh = B/Kerh \oplus Y/Kerh$

such that $B/Kerh \cong A$ and $Y/Kerh \cong S$.

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$(ii) \Rightarrow (i)$ Let $M = Y \oplus X$ for some Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.

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By hypothesis, we get that $X \cap N$ is DM in X and so M/N has a projective I–cover.

Theorem

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$i)$ every finitely presented left R–module has a projective I–cover.

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If every simple factor module of M has a projective I–cover and IM is SDM in M, then for every submodule N of M, there is a decomposition $M = A \oplus B$ such that $N = A \oplus (N \cap B)$, A is projective and $N \cap B \subseteq IM$

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Then $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$ and $ii) \Rightarrow v)$;

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THANK YOU FOR YOUR ATTENTION