

A generalization of Projective Covers

Mustafa ALKAN

Akdeniz University

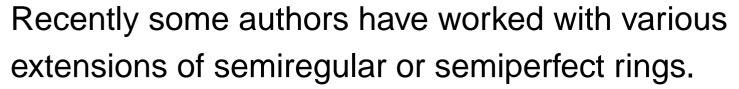
Antalya TURKEY

Join work with W.K.NICHOLSON and A.Ç.ÖZCAN



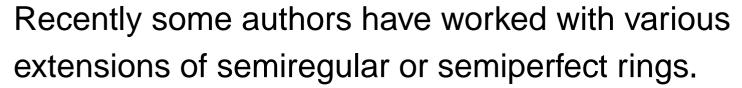
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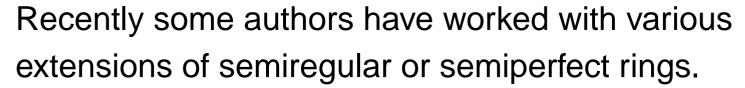
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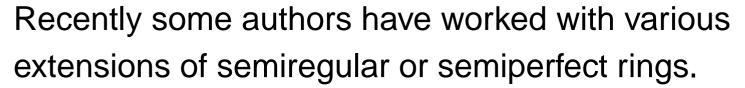
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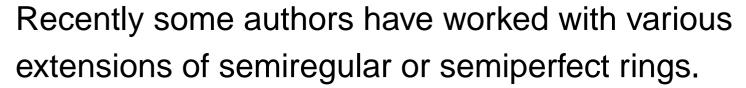
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- A module M is called δ -semiregular if for every $m \in M$ there is a decomposition $M = A \oplus B$ such that $Rm = A \oplus (Rm \cap B)$, A is projective and $Rm \cap B$ is δ -small in B.



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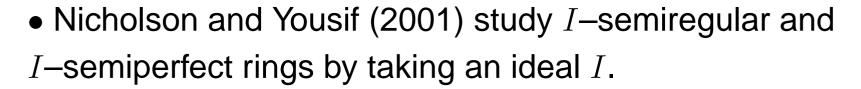


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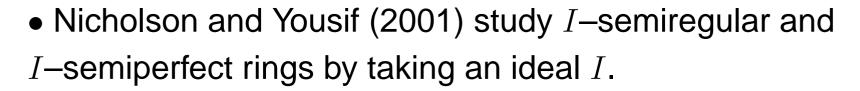




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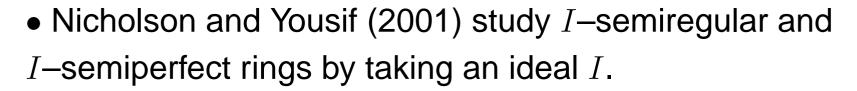
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- A left ideal I of a ring R is called strongly lifting if whenever $a^2-a\in I$, then there exists $e^2=e\in Ra$ such that $e-a\in I$.
- For an ideal I, R is I–semiregular if and only if R/I is regular and I is strongly lifting.





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Now we extend the notion of δ -small submodules and $\delta(M)$ to study a generalization of semiregular rings.









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But the converse is not true in general.









ullet M is DM for I if





Definitions

Let I be an ideal of a ring R. We say that

• M is DM for I if any submodule of IM is DM in M.





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- M is DM for I if any submodule of IM is DM in M.
- R is a left (right) DM ring for I if for any finitely generated free left (right) R-module is DM for I.



Examples

ullet Any module is DM for $Soc({}_RR)$.





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- Any module is DM for $Soc(_RR)$.
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- ullet Any finitely generated module is DM for a $\delta-$ small ideal I.





Lemma

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Projective cover

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- A pair (P, f) is called a projective I-cover of M if (P, f) is a projective I-semicover and Kerf is DM in P.





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- ullet A module has a projective 0—cover if and only if M is projective.





A module M has a projective $\delta({}_RR)$ —cover (projective J(R) —cover, resp.)



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Also we extend some well–known theorems about projective modules.



Let M have a projective I-semicover and IM=M. Then





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ullet if I is δ -small in ${}_RR$, then





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 $ii) \ M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.

Then $i) \Rightarrow ii)$,



Let M be a projective module and $N \leq M$. Consider the following conditions:

i) M/N has a projective I-cover,

 $ii) \ M = Y \oplus X$ for some submodules Y and X with $Y \subseteq N$ and $X \cap N \subseteq IM$.

Then $i) \Rightarrow ii$, if M is DM for I, then $ii) \Rightarrow i$).



$$(i) \Rightarrow (ii)$$



$$(i) \Rightarrow (ii)$$

M/N



$$(i) \Rightarrow (ii)$$

$$f:Q \longrightarrow M/N \longrightarrow 0$$
 such that $Ker\ f \subseteq IQ$ and $Kerf$ is DM in Q .



$$(i) \Rightarrow (ii)$$

M

$$\downarrow p$$

$$f:Q$$
 -

M/N

$$\longrightarrow$$
 (

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$$(i) \Rightarrow (ii)$$
 M

$$h \swarrow \qquad \downarrow p$$

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$$\begin{array}{c} (i)\Rightarrow (ii) & M \\ & h\swarrow \downarrow p \\ & f:Q & \longrightarrow M/N & \longrightarrow 0 \\ \text{such that } Ker\ f\subseteq IQ \ \text{and} \ Kerf \ \text{is} \ DM \ \text{in} \ Q. \end{array}$$

Then
$$(M)h + Kerf = Q$$
.



$$(i)\Rightarrow (ii) \qquad M$$

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$$f:Q \longrightarrow M/N \longrightarrow 0$$
 such that $Ker\ f\subseteq IQ$ and $Kerf$ is DM in Q .

Then (M)h + Kerf = Q.

We get that $Q = A \oplus K$ where $A \leq (M)h$ and $K \leq Kerf$.



Proof

$$(i)\Rightarrow (ii) \qquad M$$

$$h\swarrow \downarrow p$$

$$f:Q \longrightarrow M/N \longrightarrow 0$$
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We get that $Q = A \oplus K$ where $A \leq (M)h$ and $K \leq Kerf$.

Then $(M)h = A \oplus S$ where $S = K \cap (M)h \subseteq Kerf$.



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There is a decomposition $M/Kerh = B/Kerh \oplus Y/Kerh$ such that $B/Kerh \cong A$ and $Y/Kerh \cong S$.



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There is a decomposition $M/Kerh = B/Kerh \oplus Y/Kerh$ such that $B/Kerh \cong A$ and $Y/Kerh \cong S$.







Then $B=X\oplus Kerh$ and so M=B+Y=X+Y.

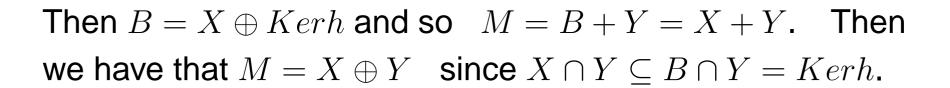




Then $B=X\oplus Kerh$ and so M=B+Y=X+Y. Then we have that $M=X\oplus Y$







By using the commutative diagram and the projectivity of X,



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$$(ii) \Rightarrow (i)$$



By using the commutative diagram and the projectivity of X, it can be proved that $Y \subseteq N$ and $N \cap X \subseteq IM$.

 $(ii)\Rightarrow (i)$ Let $M=Y\oplus X$ for some Y and X with $Y\subseteq N$ and $X\cap N\subseteq IM$.



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 $(ii)\Rightarrow (i)$ Let $M=Y\oplus X$ for some Y and X with $Y\subseteq N$ and $X\cap N\subseteq IM$.

By hypothesis, we get that $X \cap N$ is DM in X and so



By using the commutative diagram and the projectivity of X, it can be proved that $Y \subseteq N$ and $N \cap X \subseteq IM$.

 $(ii)\Rightarrow (i)$ Let $M=Y\oplus X$ for some Y and X with $Y\subseteq N$ and $X\cap N\subseteq IM$.

By hypothesis, we get that $X \cap N$ is DM in X and so M/N has a projective I–cover.









Consider the following conditions;

i) every finitely presented left R-module has a projective I-cover.



- i) every finitely presented left R-module has a projective I-cover.
- ii) for every finitely generated left ideal K of R, R/K has a projective I-cover.

Then
$$i) \Rightarrow ii$$



- i) every finitely presented left R-module has a projective I-cover.
- ii) for every finitely generated left ideal K of R, R/K has a projective I—cover.
- iii) every cyclically presented left R-module has a projective I-cover.

Then
$$i) \Rightarrow ii) \Rightarrow iii)$$



- i) every finitely presented left R-module has a projective I-cover.
- ii) for every finitely generated left ideal K of R, R/K has a projective I—cover.
- iii) every cyclically presented left R-module has a projective I-cover.
- iv) R is I-semiregular.

Then
$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv$$



- i) every finitely presented left R-module has a projective I-cover.
- ii) for every finitely generated left ideal K of R, R/K has a projective I—cover.
- iii) every cyclically presented left R-module has a projective I-cover.
- iv) R is I-semiregular.

Then
$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv$$

if R is a left DM ring for I , then $iv) \Rightarrow i$).









The following are equivalent for a ring R.

i) R is $Z(_RR)$ —semiregular.





- i) R is $Z(_RR)$ —semiregular.
- ii) Every cyclically presented left R-module has a projective $Z(_RR)$ -cover.



Corollary

- i) R is $Z(_RR)$ —semiregular.
- ii) Every cyclically presented left R-module has a projective $Z(_RR)$ -cover.
- iii) For every finitely generated left ideal K of R, R/K has a projective $Z(_RR)$ —cover.



Corollary

- i) R is $Z(_RR)$ —semiregular.
- ii) Every cyclically presented left R-module has a projective $Z(_RR)$ -cover.
- iii) For every finitely generated left ideal K of R, R/K has a projective $Z(_RR)$ —cover.
- iv) Every finitely presented left R-module has a projective $Z({}_RR)$ -cover.





Let I be an ideal of a ring R such that $I \subseteq \delta(RR)$.









Let I be an ideal of a ring R such that $I \subseteq \delta(RR)$. Then The following are equivalent for a ring R.

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- i) R is I—semiregular.
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Corollary

- i) R is I—semiregular.
- ii) Every cyclically presented left R-module has a projective I-cover.
- iii) For every finitely generated left ideal K of R, R/K has a projective I—cover.
- iv) Every finitely presented left R-module has a projective I-cover.





Let M be a finitely generated projective R-module.





Let M be a finitely generated projective R-module.

If every simple factor module of M has a projective I-cover and IM is SDM in M, then



Let M be a finitely generated projective R-module.

If every simple factor module of M has a projective I-cover and IM is SDM in M, then for every submodule N of M, there is a decomposition $M=A\oplus B$ such that $N=A\oplus (N\cap B), A$ is projective and $N\cap B\subseteq IM$









Consider the following conditions:

i) every factor module of a finitely generated projective left R-module has a projective I-cover



- i) every factor module of a finitely generated projective left R-module has a projective I-cover
- ii) every factor module of $_RR$ has a projective I-cover,

Then
$$i) \Rightarrow ii$$



- i) every factor module of a finitely generated projective left R-module has a projective I-cover ii) every factor module of $_RR$ has a projective I-cover,
- iii) for every countably generated left ideal L of R, R/L has a projective I—cover,

Then
$$i) \Rightarrow ii) \Rightarrow iii)$$



- i) every factor module of a finitely generated projective left $R{\operatorname{\mathsf{--module}}}$ has a projective $I{\operatorname{\mathsf{--cover}}}$
- ii) every factor module of $_RR$ has a projective I-cover,
- iii) for every countably generated left ideal L of R, R/L
- has a projective *I*-cover,
- iv) R is I—semiperfect,

Then
$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv$$



- i) every factor module of a finitely generated projective left R-module has a projective I-cover
- ii) every factor module of $_RR$ has a projective I-cover,
- iii) for every countably generated left ideal L of R, R/L has a projective I-cover,
- iv) R is I—semiperfect,
- v) every simple factor module of $_RR$ has a projective $I{\operatorname{-cover}}.$

Then
$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$$
 and $ii) \Rightarrow v)$;



- i) every factor module of a finitely generated projective left R-module has a projective I-cover
- ii) every factor module of $_RR$ has a projective I-cover,
- iii) for every countably generated left ideal L of R, R/L has a projective I—cover,
- iv) R is I—semiperfect,
- v) every simple factor module of $_RR$ has a projective I-cover.

Then
$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$$
 and $ii) \Rightarrow v)$; if R is a left DM ring for I then $iv) \Rightarrow i)$; if I is SDM in R then $v) \Rightarrow iv$







Let I be a strongly lifting ideal of a ring R. Then the following are equivalent;

i) R is I—semiperfect.





- i) R is I—semiperfect.
- ii) R/I is semisimple.





- i) R is I—semiperfect.
- ii) R/I is semisimple.
- iii) Every finitely generated left module has a projective I-semicover.



- i) R is I—semiperfect.
- ii) R/I is semisimple.
- iii) Every finitely generated left module has a projective I-semicover.
- iv) Every factor module of $_RR$ has a projective I-semicover.



- i) R is I—semiperfect.
- ii) R/I is semisimple.
- iii) Every finitely generated left module has a projective I-semicover.
- iv) Every factor module of $_RR$ has a projective $I{\operatorname{\!--semicover}}.$
- v) Every simple factor module of $_RR$ has a projective I-semicover.





THANK YOU FOR YOUR ATTENTION

