Commutativity in Leavitt Path Algebras

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(Joint Work with Gonzalo Aranda Pino)

American University in Cairo

Noncommutative rings and their applications
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Definition of a Leavitt path algebra

Given a field K and a directed graph E with vertices in E^0 and edges in E^1, and functions $r, s : E^1 \longrightarrow E^0$, the **Leavitt path algebra** $L(E)$ is the K-algebra generated by $\{v | v \in E^0\}$ and a set of variables $\{e, e^* | e \in E^1\}$ which satisfy:

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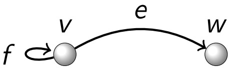
- $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every $v \in E^0$ that is not a sink.

Example



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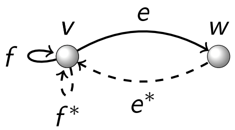

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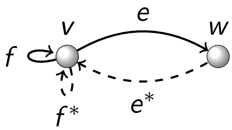

- $s(e) = v$ (so $r(e^*) = v$)
- $r(e) = w$ (so $s(e^*) = w$)

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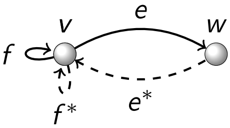

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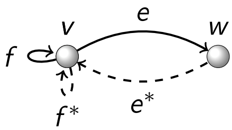

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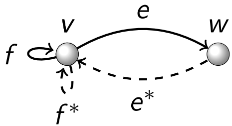

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- $ee^* + ff^* = v$

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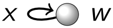
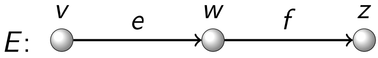


$$L(E) \cong K[x, x^{-1}]$$

Example



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$$E: \bullet^{v} \xrightarrow{e} \bullet^{w} \xrightarrow{f} \bullet^{z} \qquad L(E) \cong M_3(K)$$

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Finite-dimensional Leavitt path algebras

Theorem (Abrams, Aranda Pino, Siles Molina, 2006)

$L(E)$ is a finite-dimensional K-algebra if and only if E is a finite and acyclic graph.

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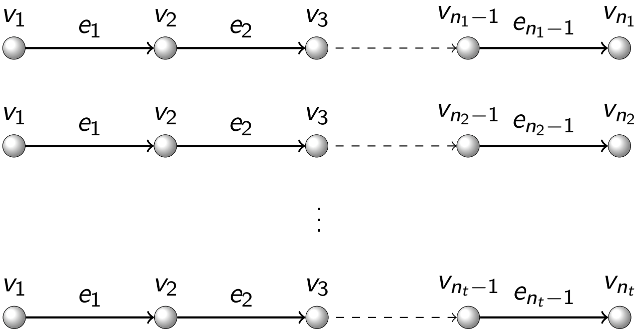
$L(E)$ is a finite-dimensional K-algebra if and only if E is a finite and acyclic graph.

The only finite-dimensional K-algebras which arise as $L(E)$ for a graph E are isomorphic to algebras of the form

$$\bigoplus_{i=1}^{t} M_{n_i}(K).$$

Finite-dimensional Leavitt path algebras



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Not unique:

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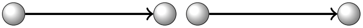
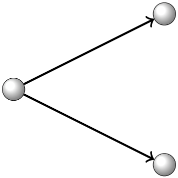
E:  $L(E) \cong M_2(K) \bigoplus M_2(K)$



F: $L(F) \cong M_2(K) \bigoplus M_2(K)$

Question:

Given any $\bigoplus_{i=1}^{t} M_{n_i}(K)$ does there exist a *connected* graph E so that

$$L(E) \cong \bigoplus_{i=1}^{t} M_{n_i}(K)?$$

Definition

If $E_1, \ldots, E_n$ is a collection of oriented line graphs (as below), then the *comet-tail graph* $G = \bigvee_{i=1}^{t} E_i$ is the graph obtained by identifying the sources of the E_i.

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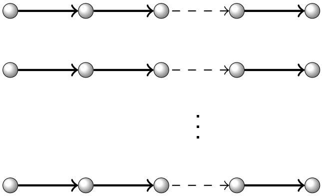

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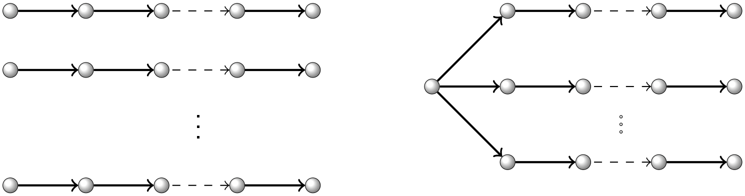

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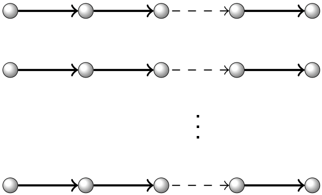

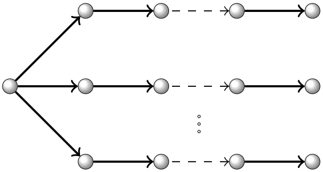

$$\bigoplus_{i=1}^{t} L(E_i) \cong L\left(\bigvee_{i=1}^{t} E_i\right)$$

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If E and F are line graphs then, by identifying the root source of E with the top source of F, we produce a new graph, which we denote by $E \bigwedge F$.

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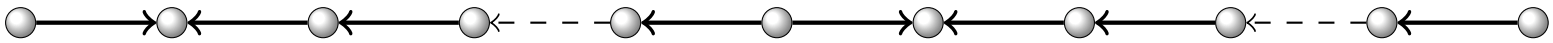

For *basic line graphs* $L\,(E \bigwedge F) \cong L(E) \oplus L(F) \cong M_n(K) \oplus M_m(K)$.

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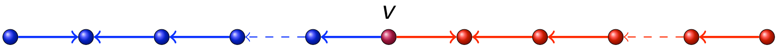

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Almost Disjoint Union

Definition

Let E be a graph. Given a collection $\{E_i\}_{i\in I}$ of subgraphs of E, we say that E is an *almost disjoint union* of $\{E_i\}_{i\in I}$ if the following conditions are satisfied:

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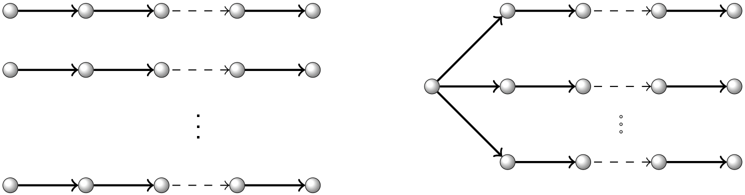
In this situation we will write $E = \coprod_{i\in I} E_i$.

Almost Disjoint Union

Both $\bigwedge$ and $\bigvee$ are special cases of an almost disjoint union.

$$\bigvee_{i=1}^{t} E_i$$



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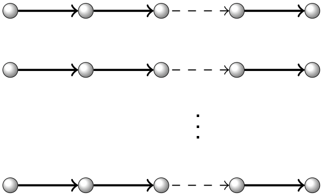

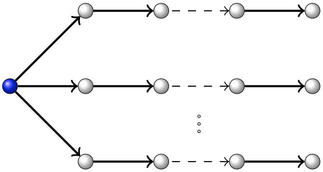

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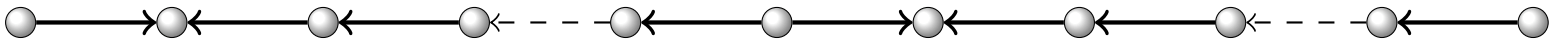

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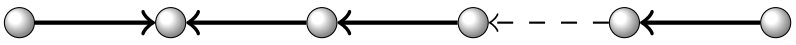
E: 

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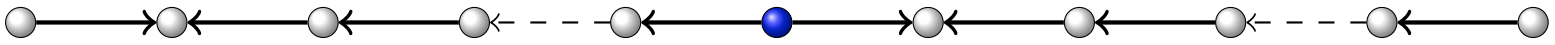

Almost Disjoint Union

Theorem

Suppose that the graph E can be written as the almost disjoint union of the subgraphs E_i. Then

$$L(E) \cong \bigoplus_{i \in I} L(E_i).$$

Finiteness Conditions

Theorem (Abrams, Aranda Pino, Perera, Siles Molina, 2009)

If E is a finite graph (i.e. $L(E)$ is unital) then $L(E)$ is one-sided artinian if and only if $L(E)$ is finite dimensional.

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The only finite-dimensional K-algebras which arise as $L(E)$ for a graph E are of the form

$$\bigoplus_{i=1}^{t} M_{n_i}(K).$$

Finiteness Conditions

Theorem (Abrams, Aranda Pino, Perera, Siles Molina, 2009)

Given a row-finite graph E, we have that $L(E)$ is locally one-sided artinian if and only if

$$L(E) \cong \bigoplus_{i \in I} M_{n_i}(K),$$

where I is countable and $n_i \in \mathbb{N} \cup \{\infty\}$.

Finiteness Conditions

Theorem (Abrams, Aranda Pino, Perera, Siles Molina, 2009)

Given a graph E, we have that $L(E)$ is locally one-sided noetherian if and only if

$$L(E) \cong \bigoplus_{i \in I_1} M_{n_i}(K) \oplus \bigoplus_{j \in I_2} M_{m_j}(K[x, x^{-1}]),$$

where I_1 and I_2 are countable and $n_i, m_j \in \mathbb{N} \cup \{\infty\}$.

Finiteness Conditions

Property of $L(E)$	Isomorphism Class	$Z(L(E))$
one-sided artinian finite-dimensional	$\bigoplus_{i=1}^{n} M_{m_i}(K)$	$\bigoplus_{i=1}^{n} K$
locally one-sided artinian semisimple	$\bigoplus_{i \in I} M_{n_i}(K)$	$\bigoplus_{i \in I' \subseteq I} K$
one-sided noetherian	$\bigoplus_{i=1}^{l} M_{m_i}(K[x, x^{-1}]) \oplus \left(\bigoplus_{i=l+1}^{l+l'} M_{m_i}(K) \right)$	$\bigoplus_{i=1}^{l} K[x, x^{-1}] \oplus \left(\bigoplus_{j=1}^{l'} K \right)$

Simplicity

Definition

- Given $w, v \in E^0$, we say $v \leq w$ if $v = w$ or there is a path μ with $s(\mu) = v$ and $r(\mu) = w$.
- A subset H of E^0 is **hereditary** if $v \leq w$ and $v \in H$ imply $w \in H$.

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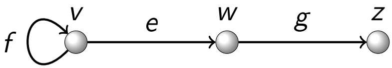
- A subset H of E^0 is **hereditary** if $v \leq w$ and $v \in H$ imply $w \in H$.
- A hereditary set is **saturated** if whenever $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ we have that $v \in H$.

Non-Simplicity

Example



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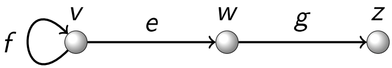

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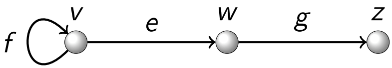

$S = \{w, z\}$ is hereditary and saturated.

$T = \{v, w\}$ is saturated but not hereditary.

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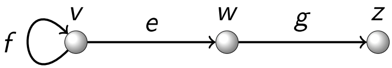

$S = \{w, z\}$ is hereditary and saturated.

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$U = \{z\}$ is hereditary but not saturated.

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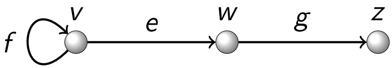

$S = \{w, z\}$ is hereditary and saturated.

$T = \{v, w\}$ is saturated but not hereditary.

$U = \{z\}$ is hereditary but not saturated.

$V = \{v, z\}$ is neither hereditary nor saturated.

Simplicity

Theorem (Abrams, Aranda Pino, (2005) 2008)

Let E be a (row-finite) graph. Then $L(E)$ is simple if and only if E satisfies the following conditions:

- *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0, and*
- *every cycle in E has an exit.*

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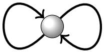

Center of a Simple Leavitt Path Algebra

Theorem

If E is a connected graph and $L(E)$ is a simple algebra then $Z = Z(L(E)) \cong K$ if $L(E)$ is unital and 0 otherwise.

Definition

A graph E satisfies *Condition (K)* if for each vertex v on a closed simple path there exists at least two distinct closed simple paths based at v.

Exchange Leavitt Path Algebras

Theorem (Aranda Pino, Pardo, Siles Molina, 2006)

A graph E satisfies Condition (K) if and only if $L(E)$ is an exchange ring.

Center of an Exchange Leavitt Path Algebra

Theorem

If $L(E)$ is a unital exchange Leavitt path algebra, then there exists a K-algebra isomorphism $Z(L(E)) \cong \bigoplus_{i=1}^{m} K$ for some $m \geq 1$.

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Let E be a graph. The following conditions are equivalent.

(i) $L(E)$ *is commutative.*

(ii) $L(E) \cong \bigoplus_{i \in I_1} K \oplus \bigoplus_{i \in I_2} K[x, x^{-1}]$, *where* $|I_1|, |I_2| \leq \aleph_0$.

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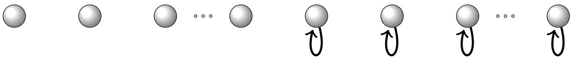
(iii) $E = \coprod_{i\in I} E_i$, *where* $|I| \leq \aleph_0$ *and each subgraph* E_i *is either an isolated vertex or an isolated loop.*



Useful Theorem

Theorem (Aranda Pino, Martín Barquero, Martín González, Siles Molina, 2008)

For every nonzero element $x \in L(E)$ *there exist* $\mu_1, \ldots, \mu_r$, $\nu_1 \ldots, \nu_s \in E^0 \cup E^1 \cup (E^1)^*$ *so that one of the following hold.*

- $\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ *is a nonzero element of* Kv, *for some* $v \in E^0$
- *there exist a vertex* w *and a cycle without exits* c *based at* w *such that* $\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ *is a nonzero element in*

$$wL(E)w = \{ \sum_{i=-m}^{n} k_i c^i \text{ for } m, n \in \mathbb{N} \text{ and } k_i \in K\}.$$

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Corollary

If $L(E)$ is simple and $x \in Z(L(E))$ then there exist (real) paths p and q, a vertex v, and a nonzero element $k \in K$ so that

$$pq^*x = kv.$$

Theorem

If E is a connected graph and $L(E)$ is a simple algebra then $Z = Z(L(E)) \cong K$ if $L(E)$ is unital and 0 otherwise.

Center of a Simple Leavitt Path Algebra

Theorem

If E is a connected graph and $L(E)$ is a simple algebra then $Z = Z(L(E)) \cong K$ if $L(E)$ is unital and 0 otherwise.

Lemma

If $\sum k_v v \in Z$ for $k_v \in K$ and $v \in E^0$ then $k_v = k_w$ for each v and w.

Proof Outline

- Suppose x is a nonzero element of Z. Then we have $pq^*x = kv$, so $p^*pxq^*q = kp^*vq$ and thus $r(p)x = kp^*q$.

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- $\cup \Theta_n = \mathrm{HSC}(r(p)) = E^0$.

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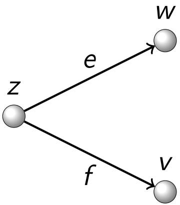
Lemma $\Rightarrow k = 0$ and thus $Z = 0$.

Example



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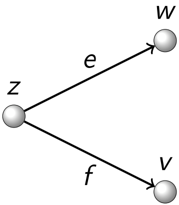

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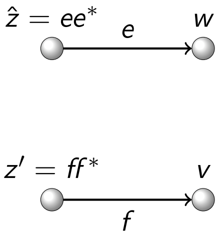

Center: $\{a(ee^* + w) + b(ff^* + v) : a, b \in K\} \cong K \oplus K$