Noncommutative rings

and their applications

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Actions of Lie superalgebras on semi-prime algebras with central invariants

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Throughout we let $\mathbb{K}$ be a field of characteristic 0.
All vector spaces are assumed to be over $\mathbb{K}$.

A vector space $L = L_0 \oplus L_1$ graded by $\mathbb{Z}_2$ together with a bilinear map $[\cdot, \cdot]\colon L \times L \to L$ is called a <u>Lie superalgebra</u> if:

1. $[L_i, L_j] \subseteq L_{i+j}$
2. $[x, y] = -(-1)^{ij}[y, x]$
3. $(-1)^{ik}[x, [y, z]] + (-1)^{ji}[y, [z, x]] + (-1)^{kj}[z, [x, y]] = 0$

for all $i, j, k \in \mathbb{Z}_2$, $x \in L_i$, $y \in L_j$ and $z \in L_k$.

Example.

Let $R = R_0 \oplus R_1$ be an associative algebra graded by $\mathbb{Z}_2$. Putting

$$[x,y] = \begin{cases} xy + yx, & \text{if } x, y \in R_1 \\ xy - yx, & \text{otherwise} \end{cases}$$

we obtain a Lie superalgebra.

Let σ be an automorphism of order 2 of an algebra R and let

$$\mathfrak{D}_0 = \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \delta(xy) = \delta(x)y + x\delta(y)$$
$$\delta\sigma(x) = \sigma\delta(x) \text{ for all } x, y \in R\}$$

$$\mathfrak{D}_1 = \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \delta(xy) = \delta(x)y + \sigma(x)\delta(y)$$
$$\delta\sigma(x) = -\sigma\delta(x) \text{ for all } x, y \in R\}.$$

Then $\mathfrak{Der}_\sigma(R) = \mathfrak{D}_0 \oplus \mathfrak{D}_1$ is a Lie superalgebra via

$$[\delta, \partial] = \begin{cases} \delta\partial + \partial\delta, & \text{if } \delta, \partial \in \mathfrak{D}_1 \\ \delta\partial - \partial\delta, & \text{otherwise} \end{cases}$$

The elements of $\mathfrak{D}_0$ and $\mathfrak{D}_1$ are called <u>derivations</u> and <u>superderivations</u> respectively.

For any elements $a, b \in R$, by ad_a and ∂_b we denote the inner derivation adjoint to a and the inner superderivation adjoint to b respectively, i.e.

$$\mathrm{ad}_a(x) = ax - xa \text{ and } \partial_b(x) = bx - \sigma(x)b.$$

The automorphism σ induces a grading $R = R_0 \oplus R_1$ by $\mathbb{Z}_2$, where

$$\begin{aligned} R_0 =& \{x \in R \mid \sigma(x) = x\} \\ R_1 =& \{x \in R \mid \sigma(x) = -x\}. \end{aligned}$$

Observe that

$$\begin{aligned} \mathrm{ad}_a\sigma = \sigma\mathrm{ad}_a \Leftrightarrow a \in R_0 \\ \partial_b\sigma = -\sigma\partial_b \Leftrightarrow b \in R_1. \end{aligned}$$

Let

$$\mathfrak{I}_0 = \{\mathrm{ad}_a \mid a \in R_0\} \text{ and } \mathfrak{I}_0 = \{\partial_b \mid b \in R_1\}.$$

Then $\mathfrak{Inn}_\sigma(R) = \mathfrak{I}_0 \oplus \mathfrak{I}_1$ is a Lie subsuperalgebra of $\mathfrak{Der}_\sigma(R)$.

We say that a Lie superalgebra $L = L_0 \oplus L_1$ acts on the algebra R if there is a homomorphism of Lie superalgebras $\psi\colon L \to \mathfrak{Der}_\sigma(R)$ satisfying $\psi(L_i) \subseteq \mathfrak{D}_i$, for all $i \in \mathbb{Z}_2$.

To simplify notation, we assume that $L \subseteq \mathfrak{Der}_\sigma(R)$ identifying the elements of L with their images under ψ.

The subalgebra of invariants R^L is defined as

$$\{x \in R \mid \delta(x) = 0 \text{ for all } \delta \in L\}.$$

Theorem (J. Bergen, P. Grzeszczuk, 1996).

Let R be a semiprime algebra over a field $\mathbb{K}$ of characteristic 0 and let L be a finite dimensional nilpotent Lie algebra which acts on R as algebraic derivations. If $R^L \subseteq \mathcal{Z}(R)$ then R is commutative and the action of L on R is trivial.

Example (J. Bergen, P. Grzeszczuk, 1996).

Let $R = M_2(\mathbb{K})$ be the algebra of 2×2 matrices over $\mathbb{K}$. Let σ be the inner automorphism of order 2 of R induced by the diagonal matrix $a = \mathrm{diag}(1, -1)$ and let ∂_{b_1} and ∂_{b_2} be the inner superderivations of R induced by

$$b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in R_1 \text{ and } b_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in R_1,$$

respectively. The superderivations ∂_{b_1} and ∂_{b_2} span an abelian Lie superalgebra $L = L_0 \oplus L_1$ where $L_0 = 0$ and $L_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{b_1}, \partial_{b_2}\}$. The subalgebra of invariants $R^L = \mathbb{K}$.

Question.

If R is a semiprime algebra over a field $\mathbb{K}$ of characteristic 0 acted on by a finite dimensional nilpotent Lie superalgebra $L = L_0 \oplus L_1$ of algebraic derivations and algebraic superderivations, and $R^L \subseteq \mathcal{Z}(R)$, must the Lie algebra L_0 act trivially on R ?

Example (P. Grzeszczuk, M.H., 2009).

Under the notations of the Bergen-Grzeszczuk Example, let $\widetilde{R} = M_2(R)$ be the algebra of 2×2 matrices over R. Let $\widetilde{\sigma}$ be the inner automorphism of order 2 of $\widetilde{R}$ induced by the diagonal matrix $\widetilde{a} = \mathrm{diag}(a, a)$ and let $\partial_{\widetilde{b_1}}, \ldots, \partial_{\widetilde{b_4}}$ be the inner superderivations of $\widetilde{R}$ induced by

$$\widetilde{b_1} = \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} \in \widetilde{R}_1, \quad \widetilde{b_2} = \begin{pmatrix} b_2 & 0 \\ 0 & -b_2 \end{pmatrix} \in \widetilde{R}_1,$$
$$\widetilde{b_3} = \begin{pmatrix} 0 & b_2 \\ -b_2 & 0 \end{pmatrix} \in \widetilde{R}_1, \quad \widetilde{b_4} = \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix} \in \widetilde{R}_1,$$

respectively. The superderivations $\partial_{\widetilde{b_1}}, \ldots, \partial_{\widetilde{b_4}}$ span an abelian Lie superalgebra $\widetilde{L} = \widetilde{L}_0 \oplus \widetilde{L}_1$ where $\widetilde{L}_0 = 0$ and $\widetilde{L}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{\widetilde{b_1}}, \ldots, \partial_{\widetilde{b_4}}\}$. The subalgebra of invariants $\widetilde{R}^{\widetilde{L}} = \mathbb{K}$.

Finally, let $\boldsymbol{R} = M_2(\widetilde{R})$ be the algebra of 2×2 matrices over $\widetilde{R}$ and let $\boldsymbol{\sigma}$ be the inner automorphism of order 2 of $\boldsymbol{R}$ induced by the diagonal matrix $\operatorname{diag}(\widetilde{a}, \widetilde{a})$. Put

$$A_1 = \begin{pmatrix} 0 & \widetilde{a_1} \\ -\widetilde{a_1} & 0 \end{pmatrix} \in \boldsymbol{R}_0 \text{ and } C_1 = \begin{pmatrix} 0 & \widetilde{a_1} \\ 0 & 0 \end{pmatrix} \in \boldsymbol{R}_0,$$

where $\widetilde{a_1} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in \widetilde{R}_0$,

$$A_2 = \begin{pmatrix} 0 & \widetilde{a_2} + 1 \\ -\widetilde{a_2} + 1 & 0 \end{pmatrix} \in \boldsymbol{R}_0 \text{ and } C_2 = \begin{pmatrix} 0 & \widetilde{a_2} + 1 \\ 0 & 0 \end{pmatrix} \in \boldsymbol{R}_0,$$

where $\widetilde{a_2} = \begin{pmatrix} 0 & b_1 b_2 \\ b_1 b_2 & 0 \end{pmatrix} \in \widetilde{R}_0$,

$$A_3 = \begin{pmatrix} \widetilde{a_3} - \widetilde{a_1} & 0 \\ 0 & \widetilde{a_3} + \widetilde{a_1} \end{pmatrix} \in \boldsymbol{R}_0,$$

where $\widetilde{a_3} = \begin{pmatrix} b_1 b_2 & b_1 b_2 \\ -b_1 b_2 & -b_1 b_2 \end{pmatrix} \in \widetilde{R}_0$,

$$B_i = \begin{pmatrix} \widetilde{b_i} & 0 \\ 0 & \widetilde{b_i} \end{pmatrix} \in \boldsymbol{R}_1 \text{ and } B_4 = \begin{pmatrix} \widetilde{b_4} & 0 \\ 0 & -\widetilde{b_4} \end{pmatrix} \in \boldsymbol{R}_1$$

for $i = 1, 2, 3$,

$B_5 = \begin{pmatrix} 0 & \widetilde{d_5} \\ \widetilde{b_5} & 0 \end{pmatrix} \in \boldsymbol{R}_1$, $B_6 = \begin{pmatrix} 0 & \widetilde{b_4} \\ -\widetilde{b_4} & 0 \end{pmatrix} \in \boldsymbol{R}_1$ and $B_7 = \begin{pmatrix} 0 & \widetilde{b_4} \\ \widetilde{b_4} & 0 \end{pmatrix} \in \boldsymbol{R}_1$,

where $\widetilde{d_5} = \begin{pmatrix} b_1 + b_2 & b_1 + b_2 \\ -b_1 - b_2 & -b_1 - b_2 \end{pmatrix}$, $\widetilde{b_5} = \begin{pmatrix} -b_1 + b_2 & -b_1 + b_2 \\ b_1 - b_2 & b_1 - b_2 \end{pmatrix} \in \widetilde{R}_1$,

$$D_5 = \begin{pmatrix} 0 & \widetilde{d_5} \\ 0 & 0 \end{pmatrix} + B_7 \in \boldsymbol{R}_1 \text{ and } D_6 = \begin{pmatrix} 0 & \widetilde{b_4} \\ 0 & 0 \end{pmatrix} \in \boldsymbol{R}_1.$$

The inner derivations $\mathrm{ad}_{C_1}, \mathrm{ad}_{C_2}, \mathrm{ad}_{A_3}$ and the inner superderivations $\partial_{B_1}, \ldots, \partial_{B_4}, \partial_{D_5}, \partial_{D_6}$ span the Lie superalgebra $\boldsymbol{N} = \boldsymbol{N}_0 \oplus \boldsymbol{N}_1$ of nilpotency class 4, where

$$\boldsymbol{N}_0 = [\boldsymbol{N}_1, \boldsymbol{N}_1] = \mathrm{span}_{\mathbb{K}}\{\mathrm{ad}_{C_1}, \mathrm{ad}_{C_2}, \mathrm{ad}_{A_3}\}$$

and

$$\boldsymbol{N}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{B_1}, \ldots, \partial_{B_4}, \partial_{D_5}, \partial_{D_6}\}$$

(see Table 1). The subalgebra of invariants $\boldsymbol{R}^{\boldsymbol{N}} = \mathbb{K}$.

$[\cdot,\cdot]$	C_1	C_2	A_3	B_1	B_2	B_3	B_4	D_5	D_6
C_1	0	0	0	0	$-2D_6$	$2D_6$	0	B_2+B_3	0
C_2	0	0	0	$2D_6$	0	0	$2D_6$	$-(B_1-B_4)$	0
A_3	0	0	0	$-2(B_2+B_3)$	$-2(B_1-B_4)$	$2(B_1-B_4)$	$-2(B_2+B_3)$	0	0
B_1	0	$-2D_6$	$2(B_2+B_3)$	$2I$	0	0	0	$2C_1$	0
B_2	$2D_6$	0	$2(B_1-B_4)$	0	$-2I$	0	0	$-2C_2$	0
B_3	$-2D_6$	0	$-2(B_1-B_4)$	0	0	$2I$	0	$2C_2$	0
B_4	0	$-2D_6$	$2(B_2+B_3)$	0	0	0	$-2I$	$2C_1$	0
D_5	$-(B_2+B_3)$	B_1-B_4	0	$2C_1$	$-2C_2$	$2C_2$	$2C_1$	$2(A_3-I)$	$-I$
D_6	0	0	0	0	0	0	0	$-I$	0

TABLE 1. operation table of $\boldsymbol{N}$

The inner derivations ad_{A_1}, ad_{A_2}, ad_{A_3} and the inner superderivations $\partial_{B_1}, \ldots, \partial_{B_4}, \partial_{B_5+B_7}, \partial_{B_6}$ span the Lie superalgebra $\boldsymbol{M} = \boldsymbol{M}_0 \oplus \boldsymbol{M}_1$ of nilpotency class 6, where

$$\boldsymbol{M}_0 = [\boldsymbol{M}_1, \boldsymbol{M}_1] = \mathrm{span}_{\mathbb{K}}\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \mathrm{ad}_{A_3}\}$$

and

$$\boldsymbol{M}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{B_1}, \ldots, \partial_{B_4}, \partial_{B_5+B_7}, \partial_{B_6}\}$$

(see Table 2). The subalgebra of invariants $\boldsymbol{R}^{\boldsymbol{M}} = \mathbb{K}$.

Finally, observe also that $\boldsymbol{M}$ is a subalgebra of a nilpotent Lie superalgebra $\boldsymbol{L} = \boldsymbol{L}_0 \oplus \boldsymbol{L}_1$ of nilpotency class 6, where

$$\boldsymbol{L}_0 = [\boldsymbol{L}_1, \boldsymbol{L}_1] = \mathrm{span}_{\mathbb{K}}\{\mathrm{ad}_{A_1}, \mathrm{ad}_{A_2}, \mathrm{ad}_{A_3}\}$$

and

$$\boldsymbol{L}_1 = \mathrm{span}_{\mathbb{K}}\{\partial_{B_1}, \ldots, \partial_{B_4}, \partial_{B_5}, \partial_{B_6}, \partial_{B_7}\}$$

(see Table 2). Obviously, $\boldsymbol{R}^{\boldsymbol{L}} = \mathbb{K}$.

Starting with the algebra $\boldsymbol{R}$ and the Lie superalgebra $\boldsymbol{L}$, and again applying the above procedure, we can produce successive examples.

$[\cdot,\cdot]$	A_1	A_2	A_3	B_1	B_2	B_3	B_4	B_5	B_6	B_7
A_1	0	$-2A_3$	0	0	$-2B_6$	$2B_6$	0	0	$-2(B_2+B_3)$	0
A_2	$2A_3$	0	0	$2B_6$	0	0	$2B_6$	0	$2(B_1-B_4)$	0
A_3	0	0	0	$-2(B_2+B_3)$	$-2(B_1-B_4)$	$2(B_1-B_4)$	$-2(B_2+B_3)$	0	0	0
B_1	0	$-2B_6$	$2(B_2+B_3)$	$2I$	0	0	0	$2A_1$	0	0
B_2	$2B_6$	0	$2(B_1-B_4)$	0	$-2I$	0	0	$-2A_2$	0	0
B_3	$-2B_6$	0	$-2(B_1-B_4)$	0	0	$2I$	0	$2A_2$	0	0
B_4	0	$-2B_6$	$2(B_2+B_3)$	0	0	0	$-2I$	$2A_1$	0	0
B_5	0	0	0	$2A_1$	$-2A_2$	$2A_2$	$2A_1$	0	$-2A_3$	0
B_6	$2(B_2+B_3)$	$-2(B_1-B_4)$	0	0	0	0	0	$-2A_3$	$2I$	0
B_7	0	0	0	0	0	0	0	0	0	$-2I$

TABLE 2. operation table of $\boldsymbol{L}$

Theorem (P. Grzeszczuk, M.H., 2009).

Let R be a finite dimensional algebra over a field $\mathbb{K}$ of characteristic 0 and let σ be an automorphism of order 2 of R. Suppose R is σ-simple. Let $L = L_0 \oplus L_1$ be a nilpotent Lie superalgebra such that $[L_0, L_1] = 0$. If L acts on R with $R^L \subseteq \mathcal{Z}(R)$ then $L_0 = 0$.