Rings determined by the properties of cyclic modules and duals: A survey-I

S. K. Jain and Ashish K. Srivastava

Dedicated to the memory of John Dauns

Abstract. This survey article addresses to the classical questions on determining rings whose cyclic modules or proper cyclic modules have certain homological property, chain condition, or combination of such properties. This survey is almost self-contained with proofs of most of the important results and references meant for all those interested in the classical ring theory.

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1. Introduction, Definitions and Notations

It is known that if R is a PID or, more generally, a Dedekind domain then for each nonzero ideal A, R/A is self-injective, that is, R/A is injective as R/A-module, equivalently, R/A is quasi-injective as R-module. The question of classifying commutative noetherian rings R such that each proper homomorphic image is self-injective was initiated by Levy [81], and later continued by Klatt-Levy [74] without assuming the noetherian condition. Later on several authors including Ahsan, Boyle, Byrd, Courter, Cozzens, Damiano, Faith, Goel, Goodearl, Hajarnavis, Hill, Huynh, Ivanov, Jain, Koehler, Mohamed, Osofsky, Singh, Skornyakov, Smith, Srivastava, and Symonds described classes of noncommutative rings whose all cyclic modules, a proper subclass of cyclic modules, injective hulls of cyclic modules, right ideals, or a proper subclass of right ideals have properties, such as, injectivity, quasi-injectivity, continuity, quasi-continuity ($= \pi$-injectivity), complements are summands, weak-injectivity, projectivity, quasi-projectivity, noetherian, or artinian.

In this paper we shall provide a survey of results on rings R over which a family of cyclic right R-modules or injective hulls of a family of cyclic modules have a certain property, or each cyclic module has a decomposition into modules with some such properties. To reiterate, these properties include injectivity, quasi-injectivity, continuity, quasi-continuity (or π-injectivity), CS (= complements are summands), weak-injectivity, projectivity, and quasi-projectivity. We will also consider rings whose proper homomorphic images are artinian or von neumann regular. The rings determined by the properties of their right ideals will be discussed in the forthcoming sequel to this survey article.

All our rings are associative rings with identity and modules are right unital unless stated otherwise. A right R-module $E \supseteq M_R$ is called an essential extension of M if every nonzero submodule of E intersects M nontrivially. E is said to be a maximal essential extension of M if no module properly containing E can be an essential extension of M. If $E \supseteq M$ is an essential extension, we say that M is an essential submodule of E, and write $M \subseteq_e E$. A submodule L of M is called an essential closure of a submodule N of M if it is a maximal essential extension of N in M. A submodule K of M is called a complement if there exists a submodule U of M such that K is maximal with respect to the property that $K \cap U = 0$. A right R-module M is called N-injective, if every R-homomorphism from a submodule L of N to M can be lifted to an R-homomorphism from N to M. A right R-module M is called an injective module if M is N-injective for every right R-module N. By Baer's criterion, a right R-module M is injective if and only if M is R_R -injective. For every right R-module M, there exists a minimal injective module containing M, which is unique upto isomorphism, called the injective hull (or injective envelope) of M. The injective hull of M is denoted by $E(M)$. $E(M)$ is indeed a maximal essential extension of M. A ring R is called right self-injective if R is injective as a right R-module. A right R-module M is called quasi-injective if $Hom_R(-, M)$ is right exact on all short exact sequences of the form $0 \longrightarrow K \rightarrow M \longrightarrow L \longrightarrow 0$. Johnson and Wong [72] characterized quasi-injective modules as those that are fully invariant under endomorphism of their injective hulls. In other words, a module M is quasi-injective if it is M-injective. More generally, as proved by Azumaya, M is N-injective if $Hom_R(E(N), E(M))N \subset M$. Consider the following properties;

(π): For each pair of submodules M_1 and M_2 of N with $M_1 \cap M_2 = 0$, canonical projection $\pi_i : M_i \rightarrow M_1 \oplus M_2, i = 1, 2$ can be lifted to an endomorphism of M.

($C1$): Every complement submodule of M is a direct summand of M.

($C2$): If N_1 and N_2 are direct summands of M with $N_1 \cap N_2 = 0$ then $N_1 \oplus N_2$ is also a direct summand of M.

($C3$): Every submodule of M isomorphic to a direct summand of M is itself a direct summand of M.

Modules satisfying the property (π) are called π-injective modules and those satisfying (C_1) and (C_3) are called quasi-continuous modules. It is known that a module is π-injective if and only if it is quasi-continuous (see [34] and [105]). Following the definition of continuous rings due to von Neumann, modules satisfying (C_1) and (C_2) are called continuous modules. Modules satisfying (C_1) are called CS-modules [18]. CS modules are also known as extending modules (see [28]). In general, we have the following implications.

Injective $\Longrightarrow$ Quasi-injective $\Longrightarrow$ Continuous $\Longrightarrow$ Quasi-continuous $\Longrightarrow$ CS

A right R-module M is called weakly N-injective if for each right R-homomorphism $\phi : N \to E(M)$, $\phi(N) \subset X \cong M$ for some submodule X of $E(M)$. A right R-module M is called weakly injective if M is weakly N-injective for each finitely generated module N. Dual to injective modules, a right R-module P is called a projective module if for any short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence of abelian groups $0 \to Hom(P, A) \to Hom(P, B) \to Hom(P, C) \to 0$ is also short exact. Equivalently, P is a direct summand of a free module. A right R-module M is called quasi-projective if for every submodule K of M the induced sequence $Hom(M, M) \to Hom(M, M/K) \to 0$ is exact. For any term not defined here, the reader may refer to [78], [79], [105], [28] and [19].

2. Rings Whose Cyclics or Proper Cyclics are Injective

The study of noncommutative rings characterized by the properties of its cyclic modules has a long history. The first important contribution in this direction is due to Osofsky [89] who considered rings over which all cyclic modules are injective. It is clear that if each R-module is injective, then R is semisimple artinian. Osofsky showed that R is semisimple artinian by simply assuming that cyclic R-modules are injective.

We begin with the theorem of Osofsky.

Theorem 2.1. (*Osofsky,* [89]). *If each cyclic R-module is injective then R must be semisimple artinian.*

Proof. By the hypothesis each principal right ideal of R is injective and hence a direct summand of R. Therefore R is a von Neumann regular ring. Using set-theoretic arguments, Osofsky proved that R has finite uniform dimension, and hence R is semisimple artinian. Instead of giving the details of this part here, we will later give proof of a more general result due to Dung-Huynh-Wisbauer (Theorem 2.8) which would imply that under the given hypothesis, each homomorphic image of R has finite uniform dimension. □

The initial proof of Osofsky in her dissertation was quite elaborate. Later Osofsky gave a shorter proof of this theorem [90]. Indeed, Skornyakov [101] had also attempted to give a proof of this theorem but unfortunately his proof had an error (see [91]).

For commutative rings the classification of commutative noetherian rings whose proper homomorphic images are self-injective was obtained by Levy.

Theorem 2.2. (*Levy,* [81]). *Let R be a commutative noetherian ring. Then every proper homomorphic image of R is a self-injective ring if and only if*

(1) R is a Dedekind domain, or

(2) R is an artinian principal ideal ring, or

(3) R is a local ring whose maximal ideal M has composition length 2 *and satisfies $M^2 = 0$.*

Proof. Suppose that every proper homomorphic image of R is self-injective. Assume first that R is a domain. If M is a maximal ideal of R, the ring R/M^2 must be self-injective and the image $\bar{M}$ of M in R/M^2 satisfies $\bar{M}^2 = 0$. It may be deduced that R cannot have any ideals between M and M^2 (see Lemma, [81]). Now, we invoke a result of Cohen [20] which states that if a noetherian domain R has the property that for every maximal ideal M, there are no ideals between M and M^2 then R must be a Dedekind domain.

Next, we consider the case when R is not a domain. Then 0 is not a prime ideal. For each prime ideal P, by hypothesis, R/P is an injective and hence a divisible R/P-module. Thus, R/P is a field and so P is a maximal ideal. But a commutative noetherian ring in which every prime ideal is maximal must be an artinian ring (see Theorem 1, [20]). Therefore, $R = R_1 \oplus ... \oplus R_n$ where each R_i is a local artinian ring. Let M_i be the maximal ideal of R. Again we consider two cases. Suppose first that either $n > 1$, or $n = 1$ but $M_1^2 \neq 0$. Then for each i, $R/(M_i^2 + \Sigma_{j \neq i} R_j) \cong R_i/M_i^2$ is a self-injective ring. Now R/M_i^2 is a commutative self-injective ring with a maximal ideal $N = M_i/M_i^2$ such that $N^2 = 0$. Thus, by (Lemma, [81]), $0 \subseteq N \subseteq R/M_i^2$ are all of the ideals of R/M_i^2. Therefore, for any $m_i \in M_i$ with $m_i \notin M_i^2$, we have $M_i = R_i m_i + M_i^2 = R_i m_i + (J(R_i))M_i$; and Nakayama's lemma then shows that $M_i = R_i m_i$. Thus R_i is a commutative noetherian ring in which every maximal ideal is principal, therefore by Kaplansky [73], R_i must be a principal ideal ring. Hence R is an artinian principal ideal ring. Finally, suppose that R is a local artinian ring whose maximal ideal M satisfies $M^2 = 0$. We may assume that M has composition length at least 2, since otherwise R would be a principal ideal ring. Since R is artinian, M contains a minimal ideal N of R. Now, R/N is a commutative self-injective ring with a maximal ideal M/N such that $(M/N)^2 = 0$, therefore by (Lemma, [81]), $0 \subseteq M/N \subseteq R/N$ are all the ideals of R/N. Hence M has composition length 2.

Conversely, suppose that R is of type (1), (2), or (3). Observe that the proper homomorphic images of all three types of rings are all artinian principal ideal rings and an artinian principal ideal ring is a direct sum of rings S which have exactly one composition series $S \supset Sm \supset Sm^2 \supset ... \supset Sm^t = 0$ (see [73]). Therefore, it suffices to show that this ring S is self-injective. Let f be a homomorphism of Sm^i into S. Then the composition length of $f(Sm^i)$ cannot exceed that of Sm^i. Since the above composition series contains all the ideals of S, we have $f(Sm^i) \subseteq Sm^i$. Consequently, $f(m^i) = m^i x$ for some $x \in S$. The map $r \to rx$ ($r \in S$) is then the extension of f to an endomorphism of S, showing that S is self-injective. □

Levy gave example of a non-noetherian domain whose proper homomorphic images are self-injective.

Example. (Levy, [81]). Let F be a field and x an indeterminate; and let W be the family of all well-ordered sets $\{i\}$ of nonnegative real numbers, the order relation being the natural order of the real numbers. Let $R = \{\Sigma_{j\in\{i\}} a_j x^j : a_j \in F, \{i\} \in W\}$. Note that every element of R whose constant term is nonzero is invertible in R. It follows that every nonzero element of R has the form $x^b u$ where u is invertible in R. This implies that R has only two types of nonzero ideals: The principal ideals (x^b), and those of the form $(x^{>b}) = \{x^c u : c > b$ and u is invertible or zero$\}$. Let $S = R/I$ where $I \neq 0$. Then S can be considered as the collection of formal power series $\Sigma_{j\in\{i\}} a_j x^j$ with $a_j \in F, \{i\} \in W$ and

(I) $y^b = 0$ if $I = (x^b)$, or

(II) $y^c = 0$ for $c > b$ if $I = (x^{>b})$. Observe that for $c \leq b$, we have
(A1): If $I = (x^b)$, then $ann(y^c) = (y^{b-c})$ and $ann(y^{>c}) = (y^{b-c})$.
(A1): If $I = (x^{>b})$, then $ann(y^c) = (y^{>(b-c)})$ and $ann(y^{>c}) = (y^{b-c})$.
From (A1) and (A2) it follows that whether S is of type (I) or (II), the principal ideals of S satisfy $ann(ann(y^c)) = (y^c)$.

To see that S is self-injective, let f be an S-homomorphism from an ideal K of S to S. If $K = (y^c)$ then $ann(K)f(K) = f(0) = 0$ so that $f(K) \subseteq ann(ann(K)) = K$. Hence $f(y^c) = y^c p$ for some $p \in S$. Thus f can be extended to the endomorphism $s \to sp$ of S.

Next, let $K = (y^{>c})$, and choose an infinite sequence $c(1) > c(2) > ...$ such that $\lim_{i\to\infty} c(i) = c$. Then $K = \cup_{i=1}^{\infty}(y^{c(i)})$. For each i the previous paragraph shows that we can choose a "power series" p_i such that $f(y^{c(i)}) = y^{c(i)}p_i$. If $j > i$ so that $y^{c(i)} = y^{c(i)-c(j)}y^{c(j)}$ the fact that f is an S-homomorphism shows that $f(y^{c(i)}) = y^{c(i)}p_j$ so that we have $p_j - p_i \in ann(y^{c(i)})$. If (A1) holds, this means that all the terms of p_i of degree $< b\tilde{N}c(i)$ are equal to the terms of the same degree of p_j while the terms of higher degree do not affect the products $y^{c(i)}p_i$ and $y^{c(i)}p_j$. A similar statement is true if (A2) holds. Thus we can assemble a single "power series" p such that $(p - p_i)y^{c(i)} = 0$ for all i (It may be verified that the collection of exponents appearing in p is well-ordered so that p is in S). Then the map $s \to sp$ extends f to an endomorphism of S. This shows that S is self-injective. Thus R is a non-noetherian domain whose every proper homomorphic image of R is self-injective.

The result of Levy was extended to noncommutative rings by Hajarnavis. Recall that a ring is called right bounded if every essential right ideal of R contains an ideal which is essential as a right ideal. A ring R is called a Dedekind prime ring if it is hereditary noetherian Asano order. Hajarnavis considered noetherian bounded prime ring such that every proper homomorphic image is a self-injective ring and proved the following.

Theorem 2.3. (*Hajarnavis,* [40]). *Let R be a noetherian, bounded, prime ring such that every proper homomorphic image of R is a self-injective ring. Then R is a Dedekind prime ring.*

Klatt and Levy later described all commutative rings, not necessarily noetherian, all of whose proper homomorphic images are self-injective. Such rings include Prufer domains.

Theorem 2.4. (*Klatt and Levy,* [74]). *A commutative ring R is pre-self-injective, that is for each non-zero ideal A, R/A is self-injective if and only if R is one of the following:*

(1) A Prufer domain such that the localization R_M for each maximal ideal M is an almost maximal rank 1 valuation domain; and every proper ideal is contained in only finitely many maximal ideals, or

(2) The finite direct sum of self-injective maximal valuation rings of rank 0*, or*

(3)An almost maximal rank 0 *valuation ring, or*

(4)A local ring whose maximal ideal M has composition length 2 *and satisfies* $M^2 = 0$.

Furthermore, finitely generated modules over pre-self-injective domains are direct sum of cyclic modules and ideals. In this connection we state results of Koethe, Cohen-Kaplansky, and Nakayama.

Theorem 2.5. (*Koethe,* [77]). *Over an artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring.*

Theorem 2.6. (*Cohen and Kaplansky,* [22]). *If R is a commutative ring such that each R-module is a direct sum of cyclic modules then R must be an artinian principal ideal ring.*

Nakayama [87] gave example of a noncommutative right artinian ring R whose each right module is a direct sum of cyclic modules but R is not a principal right ideal ring.

By Osofsky's theorem, if each cyclic R-module is injective, then each R-module has finite uniform dimension. This was extended by Osofsky-Smith [97] in the following theorem.

Theorem 2.7. (*Osofsky and Smith,* [97]). *Let R be a ring such that every cyclic right R-module is CS. Then every cyclic right R-module is a finite direct sum of uniform modules.*

We give below a proof of a more general result due to Huynh-Dung-Wisbauer. Its proof follows the same techniques as that of Osofsky-Smith [97]. We shall prove Huynh-Dung-Wisbauer's theorem in the case when M is a cyclic module with the property that each factor module is a direct sum of a CS module and a module with finite uniform dimension.

Theorem 2.8. (*Huynh, Dung and Wisbauer,* [48]). *Let M be a cyclic module such that each factor module of M is a direct sum of a CS module and a module with finite uniform dimension. Then M must have finite uniform dimension.*

Before we prove the above theorem we will prove a basic lemma which plays a key role not only in the proof of the above theorem but at several places in such problems.

Lemma 2.9. (*Huynh, Dung and Wisbauer,* [48]). *Let M be a finitely generated CS module. If M contains an infinite direct sum of nonzero submodules $N = \oplus_{\mathbb{N}} N_i$, then the factor module M/N does not have finite uniform dimension.*

Proof. Assume to the contrary that M/N has finite uniform dimension, say n. Partition $\mathbb{N}$ as a disjoint union of infinite sets $P_1, P_2, ..., P_{n+1}$ and set $U_j = \oplus_{i \in P_j} N_i$. Then $N = \oplus_{j=1}^{\infty} U_j$. Let E_j be a maximal essential extension of U_j for each $j \leq n+1$. Since M is a CS module, each E_j is a direct summand of M and hence finitely generated. This implies $E_j/U_j \neq 0$. Now, $M/N = M/(\oplus_{j=1}^{\infty} U_j)$ contains a submodule isomorphic to $E_1/U_1 \oplus E_2/U_2 \oplus ... \oplus E_{n+1}/U_{n+1}$. This yields a contradiction to our assumption that $u.\dim(M/N) = n$. Hence, the factor module M/N must have infinite uniform dimension. □

Now we are ready to give the proof of the Theorem 2.8 . Throughout this proof we will denote by A^e a maximal essential extension of a module A.

Proof. Assume to the contrary that M does not have finite uniform dimension. By hypothesis $M = M_1 \oplus M_2$ where M_1 is CS and M_2 is of finite uniform dimension. This implies $u.\dim M_1 = \infty$. Thus without loss of generality we can assume M is a CS module of infinite uniform dimension such that each factor module is a direct sum of a CS module and a module of finite uniform dimension. Let $\oplus_{i \in I} M_i$ be an infinite direct sum of submodules in M. Then it follows that for every positive integer k, $M = M_1^e \oplus ... \oplus M_k^e \oplus N_k$ for some submodule N_k of M. Note that since M is cyclic, each summand in the decomposition of M is cyclic, and hence there exists maximal submodules $U_i \subset M_i^e$. Denote the simple module M_i^e/U_i by S_i. Then we have

$$(*): \quad M/(\oplus_{i \in I} U_i) = (M_1^e \oplus ... \oplus M_k^e \oplus N_k)/(U_1 \oplus ... \oplus U_k \oplus S') \simeq S_1 \oplus ... S_k \oplus X,$$

yielding a direct sum $S = \oplus_{i \in I} S_i$ of simple modules in the factor module$M/(\oplus_i U_i)$. Set $\bar{M} = M/(\oplus_{i \in I} U_i$. By hpothesis, $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ is CS and $\bar{M}_2$ has finite uniform dimension. We can write $S = (S \cap \bar{M}_1) \oplus K$. Since $K \cap \bar{M}_1 = 0$, K is embeddable in $\bar{M}_2$. Note that $S \cap \bar{M}_1$ must be infinitely generated because S is infinitely generated. From (*) it follows that every finitely generated submodule of S, (and hence of $S \cap \bar{M}_1$) is a direct summand of $\bar{M}$and hence of $\bar{M}_1$. Let $(S \cap \bar{M}_1)^e$ denote a maximal essential extenssion of $S \cap \bar{M}_1$ in $\bar{M}_1$ and note that $\bar{M}_1$ is CS and cyclic because it is a direct summand of a cyclic module. This implies $(S \cap \bar{M}_1)^e$ is a direct summand of $\bar{M}_1$ and hence cyclic. Claim: $(S \cap \bar{M}_1)^e/(S \cap \bar{M}_1)$ is CS. We write $(S \cap \bar{M}_1)^e/(S \cap \bar{M}_1) = \bar{C} \oplus \bar{D}$, where $\bar{C} = C/(S \cap \bar{M}_1)$ is CS and

$\bar{D} = D/(S \cap \bar{M}_1)$ has finite uniform dimension. We procced to prove that $\bar{D} = 0$. Now $\bar{D} = \frac{dR+S\cap\bar{M}_1}{S\cap\bar{M}_1}$ for some element $d \in R$. By hypothesis, write $dR = L \oplus T$ where L is CS and $u.\dim T$ is finite. As $S\cap\bar{M}_1 \subset_e (S\cap\bar{M}_1)^e$, $\frac{dR+S\cap\bar{M}_1}{S\cap\bar{M}_1}$ is singular and so $dR \cap (S \cap \bar{M}_1)$ is essential in dR. This gives, $Socle(dR) \subset_e dR$. We have $Socle(dR) = Socle(T) \oplus Socle(L)$ and $Socle(dR) \subset_e dR$, so $Socle(T) \subset_e T$. But, $Socle(T)$ is a finite direct sum of simple modules and hence by (*), $Socle(T)$ is a direct summand of T. Hence, $Socle(T) = T \subset S\cap\bar{M}_1$. Moreover, $L/(L\cap(S\cap\bar{M}_1)) \cong (L + S \cap \bar{M}_1)/(S \cap \bar{M}_1) \cong (dR + S \cap \bar{M}_1)/(S \cap \bar{M}_1) = \bar{D}$ has finite uniform dimension. Therefore by Lemma 2.9, $S \cap \bar{M}_1$ is finitely generated and hence is a direct summand of $(S\cap\bar{M}_1)^e$. Thus, $L = L\cap(S\cap\bar{M}_1) \subset S\cap\bar{M}_1$ and hence $\bar{D} = 0$. Therefore, $(S \cap \bar{M}_1)^e/(S \cap \bar{M}_1)$ is CS. Now, we partition $\mathbb{N}$ as a disjoint union of infinite sets $\{P_i\}_{i\in\mathbb{N}}$. Then $S\cap\bar{M}_1 = \oplus_{j=1}^{\infty}(\oplus_{i\in P_j}(S_i\cap\bar{M}_1))$. Let $(\oplus_{i\in P_j}(S_i\cap\bar{M}_1)^e$ be a maximal essential extension of $(\oplus_{i\in P_j}(S_i \cap \bar{M}_1)$ in $(S \cap \bar{M}_1)^e$. For simplicity we denote $(\oplus_{i\in P_j}(S_i \cap \bar{M}_1)^e$ by Y_j. Let $\bar{Y}_j$ be the image of Y_j under the canonical morphism $(S\cap\bar{M}_1)^e \to (S\cap\bar{M}_1)^e/(S\cap\bar{M}_1)$. Since $(S\cap\bar{M}_1)^e/(S\cap\bar{M}_1)$ is CS, we may find a maximal essential extension $\bar{Y}$ of $\oplus_{\mathbb{N}}\bar{Y}_j$ in $(S\cap\bar{M}_1)^e/(S\cap\bar{M}_1)$ which is a direct summand of $(S \cap \bar{M}_1)^e/(S \cap \bar{M}_1)$. As $\bar{Y}$ is cyclic, there exists a cyclic submodule $Y \subset (S\cap\bar{M}_1)^e$ with $\bar{Y} = (Y+S\cap\bar{M}_1)/(S\cap\bar{M}_1)$. For each $j \in \mathbb{N}$, we have $\bar{Y}_j \subset Y + S\cap\bar{M}_1 = Y+(Y\cap(S\cap\bar{M}_1))+S\cap\bar{M}_1 = Y+(Y\cap(S\cap\bar{M}_1)\oplus V) = Y\oplus V$ for a suitable submodule V of $S\cap\bar{M}_1$. On the other hand, by (*), $Y = Y' \oplus Y''$ where Y' is CS and Y'' has finite uniform dimension. We have $Socle(Y) = Socle(Y') \oplus Socle(Y'')$ and $Socle(Y) \subset_e Y$, so $Socle(Y'') \subset_e Y''$. But, $Socle(Y'')$ is a finite direct sum of simple modules and hence by (*), $Socle(Y'')$ is a direct summand of Y''. Hence, $Socle(Y'') = Y'' \subset S\cap\bar{M}_1$, and for each $j \in \mathbb{N}$, we have $\bar{Y}_j \subset Y+S\cap\bar{M}_1 = Y'\oplus V'$ with $V' = V \oplus Y'' \subset S \cap \bar{M}_1$. Assume $Y' \cap Y_j = 0$ for some $j \in \mathbb{N}$. Then Y_j can be embedded in V'. This implies Y_j is semisimple of finite length, a contradiction. Hence $Y' \cap Y_j \neq 0$ for each $j \in \mathbb{N}$. Now take a minimal submodule W_j in every $Y' \cap Y_j$ and denote by $(\oplus_{\mathbb{N}}W_i)^e$ a maximal essential extension of $\oplus_{\mathbb{N}}W_i$ in Y'. Since Y' is CS, $(\oplus_{\mathbb{N}}W_i)^e$ is a direct summand of Y'. Then $(\oplus_{\mathbb{N}}W_i)^e$ is a cyclic module with an essential socle of infinite length and hence $(\oplus_{\mathbb{N}}W_i)^e \not\subset S \cap \bar{M}_1$. Thus, $((\oplus_{\mathbb{N}}W_i)^e + S \cap \bar{M}_1)/(S \cap \bar{M}_1) \neq 0$. We have $\oplus_{j=1}^{m}W_j \subset_e (\oplus_{\mathbb{N}}W_i)^e$. This yields $(\oplus_{j=1}^{m}W_j)\cap(\oplus_{j=1}^{m}Y_j) \subset_e (\oplus_{\mathbb{N}}W_i)^e\cap(\oplus_{j=1}^{m}Y_j)$. But as each $W_j \subset Y_j$, we get $(\oplus_{j=1}^{m}W_j) \subset_e (\oplus_{\mathbb{N}}W_i)^e \cap (\oplus_{j=1}^{m}Y_j)$. But as $(\oplus_{j=1}^{m}W_j)$ is also a direct summand of $(\oplus_{\mathbb{N}}W_i)^e \cap (\oplus_{j=1}^{m}Y_j)$, we conclude that $(\oplus_{j=1}^{m}W_j) = (\oplus_{\mathbb{N}}W_i)^e \cap (\oplus_{j=1}^{m}Y_j)$ for each $m \in \mathbb{N}$. This implies $(\oplus_{\mathbb{N}}W_i)^e \cap (\oplus_{j\in\mathbb{N}}Y_j) \subset S \cap \bar{M}_1$ which yields a contradiction to the fact that $((\oplus_{j\in\mathbb{N}}Y_j) + S \cap \bar{M}_1)/(S \cap \bar{M}_1)$ is essential in $\bar{Y}$. Therefore, M must have finite uniform dimension. □

Boyle ([10], [11]) initiated the study of right noetherian rings over which each proper cyclic module is injective. Note that this property does not hold for all Dedekind domains. Boyle called these rings right PCI rings and studied right noetherian right PCI-rings. Right PCI-rings without chain condition were studied by Faith. Later Damiano proved that they are indeed right noetherian.

Theorem 2.10. (*Boyle,* [10]). *A right noetherian right PCI-ring is either a semisimple artinian ring or a simple right hereditary domain.*

We first prove a lemma for rings R over which proper cyclic modules are quasi-injectve. Thus this lemma holds for right PCI-rings also.

Lemma 2.11. *Let R be a prime ring such that each proper cyclic right module is quasi-injective. Then R is either artinian or a right Ore domain.*

Proof. We first show that R is right nonsingular. Assume to the contrary that $Z(R_R) \neq 0$. By Theorem 2.8, R has finite right uniform dimension and therefore, there is a uniform submodule U of R_R. Consider $S = U \cap Z(R_R)$. Since R is prime and $S \neq 0$, we have $S^2 \neq 0$. Let $a \in S$ be such that $aS \neq 0$ and $0 \neq x \in S \cap ann_r(a)$. Since $x \in Z(R_R)$, $xR \ncong R$. Therefore, by hypothesis, xR is quasi-injective. Let $E(S)$ be the injective hull of S. As U is uniform, $xR \subset_e U$ and hence $xR \subset_e S$. Therefore, xR is a fully invariant submodule of $E(S)$. In particular, $S(xR) \subseteq xR$. Thus $(aS)(xR) = a(SxR) \subseteq a(xR) = 0$, while $aS \neq 0$ and $xR \neq 0$, a contradiction to the primeness of R. Therefore, R is right nonsingular. Hence, R is a right Goldie ring. If R_R is uniform the either R is a division ring or a right Ore domain. Assume that R_R is not uniform. Let $U_1 \oplus U_2 \oplus ... \oplus U_m \subset_e R_R$, where $m > 1$ and each U_i is a uniform right ideal of R. Let $0 \neq a_1 \in U_1$. Then $a_1R \ncong R$. Therefore, a_1R is quasi-injective. Since R is a prime right Goldie ring, all uniform right ideals of R are subisomorphic to each other. Hence, each U_i, $(i \geq 2)$ contains an isomorphic copy a_iR of a_1R. It follows that $A = a_1R \oplus a_2R \oplus ... \oplus a_mR$ is quasi-injective. Since A is essential in R_R, A contains a regular element b. Thus, as A is bR-injective and $bR \cong R$, A is injective. Therefore, $A = R$ and R is right self-injective. Hence R is simple artinian. □

Faith proved the following for right PCI-rings without assuming any chain condition.

Theorem 2.12. (*Faith,* [29]). *A right PCI ring R is either semisimple artinian or a simple right semi-hereditary right Ore V-domain.*

Proof. If A is any nonzero ideal, then by Osofsky theorem R/A is semisimple artinian. In particular, every nonzero prime ideal is maximal.

Suppose R is not prime. Then there exists nonzero ideals A and B such that $AB = 0 \subset P$, for any prime ideal P. This implies either A or B is contained in P. Furthermore, R/A and R/B are semisimple artinian. This means there are only finitely many prime ideals above A and B. Since every prime ideal contains either A or B, it follows that there are only finitely many prime ideals in R. This gives that the prime radical $N = P_1 \cap ... \cap P_k$ and $R/N \cong R/P_1 \times ... \times R/P_k$. By Ososfsky, R/P_i is simple artinian for each i. Thus R/N is semisimple artinian. Since N is nil, it implies $N = J(R)$. Hence R is semiperfect. Suppose R is not local. Then $R = e_1R + ... + e_nR$, where e_iR are indecomposable right ideals and by hypothesis these are injective. Then R_R is injective. Then by Osofsky theorem R is semisimple artinian. Assume now R is local. We claim $N = 0$. Else, let $0 \neq a \in N$.

If $aR \not\cong R$, then aR is injective and hance a summand of R. Since R is local, this gives $aR = R$, a contradiction because $a \in N$. Thus $aR \cong R$. This too is not possible because a is a nil element. Thus $N = 0$ and we conclude that R is semisimple artinian if it is not prime.

If R is prime then by the above lemma, R is either artinian or a right Ore domain.

Now we proceed to show that R is right semi-hereditary, that is, each finitely generated ideal of R is projective. The proof is by induction. Let $A = aR+bR$. Then $aR+bR/bR$ is cyclic. In case $aR+bR/bR$ is isomorphic to R, then $aR+bR = bR \oplus K$, where $K \cong R$. So $aR+bR$ is projective. In the other case, $aR+bR/bR$ is injective and so a direct summand of R/bR. This gives $aR+bR/bR \oplus X/bR = R/bR$, that is, $(aR+bR)+X = R$, where $(aR+bR) \cap X = bR$. Thus, $(aR+bR) \times X \cong R \times bR$, proving that $aR+bR$ is projective. By induction, we may deduce that each finitely generated right ideal of R is projective. □

The following is an example of a right and left noetherian domain which is both right and left PCI-ring. We do not know any example of a right PCI-domain which is not a left PCI-domain.

Example. (Cozzens, [23]). Let k be a universal differential field with derivation D and let $R = k[y, D]$ denote the ring of differential polynomials in the indeterminate y with coefficients in k, i.e., the additive group of $k[y, D]$ is the additive group of the ring of polynomials in the indeterminate y with coefficients in field k, and multiplication in $k[y, D]$ is defined by: $ya = ay + D(a)$ for all $a \in k$. Let $f = \Sigma_{i=1}^{n} a_i y^i \in k[y, D]$, $a_n \neq 0$. We define the degree of f, $\delta(f) = n$. Clearly we have the following: (i) $\delta(fg) = \delta(f) + \delta(g)$, (ii) for $f, g \in k[y, D]$, there exist $h, r \in k[y, D]$ such that $f = gh + r$ with $r = 0$ or $\delta(r) < \delta(g)$ (a similar algorithm holds on the left). From (ii) it follows that this ring $R = k[y, D]$ is both left and right principal ideal domain. The simple right R-modules are precisely of the form $V_\alpha = R/(y-\alpha)R$ where $\alpha \in k$. We claim that $V_\alpha = R/(y-\alpha)R$ is a divisible right R-module for all $\alpha \in k$. For this, it suffices to show that $V_\alpha(y+\beta) = V_\alpha$ for all $\alpha, \beta \in k$. Equivalently, given $h \in R$, $\delta(h) = 0$, there exist $f, g \in R$ such that $f(y+\beta)+(y+\alpha)g = h$. We shall determine $a, b \in k$ such that $a((y+\beta)+(y+\alpha)b = h$. This is equivalent to an equation of the form $D(b) + (\alpha - \beta)b = h$. Since k is a universal differential field, there exists a $b \in k$ satisfying this equation. Hence, each simple right R-module is divisible. Since R is a principal right ideal domain, this implies that each simple right R-module is injective. Therefore, R is a right V-ring. Similarly, R can be shown to be a left V-ring as well. Hence by (Corollary 10, [12]), R is both left as well as a right PCI-domain.

Osofsky provided another example of right and left PCI-ring.

Example. (Osofsky, [95]). Let F be a field of characteristic $p > 0$, and σ an endomorphism of F defined by $\sigma(\alpha) = \alpha^p$ for all $\alpha \in F$. We then form the ring of twisted polynomials with coefficients on the left, $R = F[x, \sigma]$ with $R = \{\Sigma_{i=0}^{n} \alpha_i x^i : n \in \mathbb{Z}, \alpha_i \in F\}$ under usual polynomial addition and multiplication given by the relation $x\alpha = \sigma(\alpha)x$ for all $\alpha \in F$. It may be observed that this ring R is a left and right principal ideal domain. If F is separably closed then every simple right (left) R-module is divisible and hence injective. Therefore, every proper cyclic right (left) R-module is injective, that is, R is a both left and right PCI-domain.

In [24], Cozzens and Faith asked the question if every right PCI- ring is right noetherian. This question was later answered in the affirmative by Damiano [25]. We provide a different proof that uses, in particular Osofsky-Smith theorem.

Theorem 2.13. (*Damiano,* [25]). *Let R be a right PCI ring. Then R is right noetherian and right hereditary.*

Proof. The proof follows as a special case of a result of Holston-Jain-Leroy proved recently (Proposition 9, [42]) that states for a waekly-V ring R if each cyclic right R-module is projective, CS, or noetherian then R is right noetherian.

First recall that a right PCI-ring is either semisimple artinian or a simple right Ore domain. Assume R is a right Ore domain which is not a division ring. So each right ideal is essential and socle is zero. Furthermore, each simple right R-module is isomorphic to R/M where M is a an essential maximal right ideal and thus it is not isomorphic to R. Therefore, each simple module is injective. So R is a V-ring. Also, by Osofsky-Smith theorem R/E has finite uniform dimension for every nonzero right ideal E. We proceed to prove R/E is right noetherian.

Suppose that R/E is not right noetherian. Then there is an infinite ascending sequence $a_1R \subset\ a_1R + a_2R \subset a_1R + a_2R\ +a_3R+ \subset ...$ of submodules of R/E. Put $X = \cup_n(a_1R + ... + a_nR)$. Let N_1 be a maximal submodule contained in a_1R. We denote $U_1 = a_1R$ and let S_1 be the simple right module a_1R/N_1. Since R/E is injective, we conclude that S_1 is a direct summand of X/N_1 and we can write $U_1/N_1 \oplus T_1/N_1 = X/N_1$ for some submodule T_1 of X. Clearly, T_1/N_1 is not right noetherian and hence, using the same arguments, we can find a quotient module of T_1/N_1 with a simple direct summand, i. e., we can find T_2 and U_2 submodules of T_1 such that $U_2/N_2 \oplus T_2/N_2 = T_1/N_2$. Continuing this process we can get sequences of submodules of X having the following properties: $N_1 \subset N_2 \subset ...;\ T_1 \supset T_2 \supset ...$.

Set $S_i = U_i/N_i$, $N_i \subset U_i \subset T_{i-1}$, $U_i \cap T_i = N_i, N = \cup N_i$. Let $p_i : X/N_i \to X/N$ be the canonical map. Because $U_i \cap N = N_i, p(S_i) \cong S_i$. We can show that $\sum p(S_i)$ is an infinite direct sum of submodules in $X/N \subset R/N$, a contradiction. Hence R/E is right noetherian for all nonzero right ideals E. This yields $R/Socle(R) = R$ is right noetherian as desired. Now, in view of Theorem 2.12, R must be right hereditary. □

Rings for which every singular right module is injective are called right SI rings. The right PCI and right SI conditions are equivalent for domains. Boyle and Goodearl studied the question of left-right symmetry of PCI-domains. They proved the following.

Theorem 2.14. (*Boyle and Goodearl,* [13]). *Let R be a right and left noetherian domain. Then R is a right PCI-domain if and only if R is a left PCI-domain.*

Proof. We first remark that in order to prove left PCI-domain, it suffices to assume that R is left Ore domain, because a right PCI-domain which is left Ore is left noetherian (See [24], Page 112).

Now we show that a left and right noetherian right PCI-domain is a left PCI-domain. Note each proper cyclic right as well as left module is torsion since R is right and left Ore domain. Furthermore, the functor $Hom(-, Q(R)/R)$ defines a duality between the category of finitely generated torsion right modules and the category of finitely generated torsion left modules. Because R is a right PCI-domain, each proper cyclic right module is semisimple. This gives each proper cyclic left module is semisimple. Therefore, R is a left PCI domain. □

In [25], Damiano had proved that a right PCI ring R is left PCI if and only if R is left coherent. But this is incorrect (see e.g. Remark 6.3, [60]). Jain-Lam-Leroy [60] considered twisted differential polyniomial rings over a division ring which are right PCI domains and found equivalent conditions as to when it will be left PCI. However, in general, the question of left-right symmetry of PCI domains is still open.

In [49] Huynh, Jain and Lopez-Permouth had proved that a simple ring R is right PCI if and only if every proper cyclic right R-module is quasi-injective. Barthwal-Jhingan-Kanwar showed that R is a simple right PCI domain if and only if each proper cyclic right module is continuous [4]. Huynh, Jain and Lopez-Permouth later strengthened their result by proving the following.

Theorem 2.15. (*Huynh, Jain and Lopez-Permouth,* [50]). *A simple ring R is Morita equivalent to a right PCI domain if and only if every cyclic singular right R-module is quasi-continuous.*

As a consequence of this, they also showed the following.

Theorem 2.16. (*Huynh, Jain and Lopez-Permouth,* [50]). *Let R be a simple ring. If every proper cyclic module is quasi-continuous, then R is Morita equivalent to a right PCI domain and has right uniform dimension at most* 2.

Theorem 2.17. (*Huynh, Jain and Lopez-Permouth,* [50]). *For a right V-ring R with $Soc(R_R) = 0$ the following conditions are equivalent:*

(i) Every cyclic singular right R-module is quasi-continuous.

(ii) R has a ring-direct decomposition $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, where each R_i is Morita equivalent to a right PCI domain.

Carl Faith called a ring right CSI ring [31] if for each cyclic module C, the injective hull $E(C)$ is Σ-injective. It is still an open question whether every right CSI ring is right noetherian. In the following theorem Faith showed that right CSI rings are right noetherian in some special cases.

Theorem 2.18. (*Faith,* [31]). *A right CSI ring R is right noetherian under any of the following conditions:*

(i) R is commutative.

(ii) R has only finitely many simple modules up to isomorphism, e.g. R is semilocal.

(iii) R satisfies the acc on colocal right ideals.

(iv) R or $R/J(R)$ is right Kasch.

(v) $R/J(R)$ is von Neumann regular, e.g. R is right continuous.

(vi) The injective hull of any countably generated semisimple right R-module is Σ-injective.

3. Rings whose Cyclic or Proper Cyclic Modules are Quasi-injective

In [29] Faith proposed to study the class of rings which satisfy the following condition: (P) *Every proper cyclic module is injective modulo its annihilator ideal.*

Clearly, every right PCI ring satisfies the condition (P). Commutative rings with the condition (P) are precisely the pre-self injective rings studied by Klatt and Levy [74]. It may be observed that in a right self-injective ring with the condition (P), each cyclic module is quasi-injective. These rings were studied by Ahsan [2], and Koehler ([75], [76]) among others and are called *qc*-rings. Koehler provided a complete characterization of *qc* rings in the following theorem.

Theorem 3.1. (*Koehler,* [76]). *For a ring R the following are equivalent:*

1. *Each cyclic right R-module is quasi-injective.*
2. *$R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of self-injective, rank 0, valuation, duo rings with nil radical.*
3. *Each cyclic left R-module is quasi-injective.*

Proof. If R is prime then by the Lemma 2.11 above, R is either simple artinian or Ore domain. But because R is also right self-injective, R is simple artinian in both the cases. This implies that each prime ideal is maximal. By the same argument as in Theorem 2.12, R is semiperfect. If e_iR and e_jR are indecomposable quasi-injective (hence uniform) right ideals such that $e_iR \times e_jR$ is quasi-injective and $Hom_R(e_iR, e_jR) \neq 0$, then $e_iR \cong e_jR$, and are minimal right right ideals (see Lemma 4.1 proved later). Write $R = e_1R \oplus ..... \oplus e_nR$ as a direct sum of indecomposable quasi-injective right ideals. We can group all isomorphic indecomposable right ideals as explained above, and write after renumbering, if necessary, $R = [e_1R] \oplus ... \oplus [e_kR]$ as a direct sum of ideals, where each bracket represents the sum of indecomposable right ideals isomorphic to the term in the bracket. By taking endomorphism rings of R-modules on both sides, we obtain $R = M_{n_1}(e_1Re_1) \oplus .... \oplus M_{n_k}(e_kRe_k)$, where each $M_{n_i}(e_iRe_i)$ represents $n_i \times n_i$ matrix ring over a division ring e_iRe_i. So for $n_i \geq 2$, the matrix ring $M_{n_i}(e_iRe_i)$ is simple artinian. It then follows all matrix rings in the decomposition of R are simple artinian excepting those which are local rings. It is trivial to show a right self-injective local ring with nil radical is duo. This proves (1) $\implies$ (2). That

(2) $\Longrightarrow$ (1) is straightforward. The right-left symmetry of (2) completes the proof. □

Jain, Singh, Symonds generalized the notion of right *qc*-ring and called a ring R a right $PCQI$ ring if each proper cyclic right R-module is quasi-injective. Clearly rings with the property (P) are right $PCQI$-rings. Jain et al [71] showed the following.

Theorem 3.2. (*Jain, Singh and Symonds,* [71]).

(i) A right $PCQI$ ring R is either prime or semiperfect.

(ii) If R is non-prime, non-local then R is a right $PCQI$-ring if and only if either R is qc-ring or $R = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ *where D is a division ring.*

(iii) A local right $PCQI$-ring with maximal ideal M is a right valuation ring or $M^2 = 0$ and M_R has composition length 2.

(iv) A right $PCQI$-domain is a right Ore domain.

(v) A nonlocal semiperfect right $PCQI$-ring is also a left $PCQI$-ring.

Proof. Here also, every nonzero prime ideal is maximal because if P is a prime ideal then R/P is a *qc*-ring and hence simple artinian.

Noting that for every nonzero ideal A, R/A is a *qc*-ring and has finitely many prime ideals (see the structure of *qc*-rings). By arguing as before (see Theorem 2.12) $PCQI$-ring is either prime or semiperfect. If it is prime, then it is a right Ore domain (see Lemma 2.11). If R is a nonlocal semiperfect ring, then we proceed to show that it is either semisimple artinian or 2×2 upper triangular matrix ring over a division ring. The reader is referred to Lemma 5 - Theorem 13 in ([71], pp 462-464) for its proof. For proving (iii), the reader is referred to (Theorem 14, p. 466, [71]) (iv) follows from Lemma 2.11. (v) is a consequence of the structure of nonlocal semipefect right $PCQI$-ring. □

The following example of Jain, Singh and Symonds shows that a local right $PCQI$-ring need not be a left $PCQI$-ring.

Example. (Jain, Singh and Symonds, [71]). Let F be a field which has a monomorphism $\sigma : F \to F$ such that $[F : \sigma(F)] > 2$. Take x to be an indeterminate over F. Make $V = xF$ into a right vector space over F in a natural way. Let $R = \{(\alpha, x\beta) : \alpha, \beta \in F\}$. Define

$$(\alpha_1, x\beta_1) + (\alpha_2, x\beta_2) = (\alpha_1 + \alpha_2, x\beta_1 + x\beta_2)$$

and

$$(\alpha_1, x\beta_1)(\alpha_2, x\beta_2) = (\alpha_1\alpha_2, x(\sigma(\alpha_1)\beta_2 + \beta_1\alpha_2))$$

Then R is a local ring with maximal ideal $M = \{(0, x\alpha) : \alpha \in F\}$. In fact, M is also a maximal right ideal with $M^2 = 0$ and hence R is a right $PCQI$-ring. Further, if $\{\alpha\}_{i\in I}$ is a basis of F as a vector space over $\sigma(F)$ then $M = \oplus R(0, x\alpha_i)$ is a direct sum of irreducible left modules $R(0, x\alpha_i)$. Since $|I| > 2$, R is not a left $PCQI$-ring.

In an attempt to understand rings R satisfying condition (P), Jain and Singh [69] obtained the following.

Theorem 3.3. (*Jain and Singh,* [69]). *Let R be a ring satisfying the condition property (P). Then either R is a right Ore-domain or semiperfect. Further, a semiperfect ring R satisfies the property (P) if and only if R is a right PCQI-ring.*

The question of characterizing right Ore-domains with the property (P) is still open.

4. Rings whose Cyclic or Proper Cyclic Modules are Continuous

A ring R is called, respectively, a right *cc*-ring, a right πc-ring if each cyclic module is continuous, or π-injective. Note that π-injective modules are known as quasi-continuous modules. Jain and Mohamed [64] generalized Koehler's theorem and gave the structure of *cc*-rings. But first we prove a more general result in the following lemma that holds for rings over which cyclic modules are π-injective (= quasi-continuous). The reader will come across the application of the next lemma at several places.

Lemma 4.1. *Let R be a ring over which each cyclic module is π-injective (= quasi-continuous). If e and f are indecomposable orthogonal idempotents in R such that $eRf \neq 0$, then eR and fR are isomorphic minimal right ideals of R.*

Proof. Let $0 \neq eaf \in eRf$. Now $eafR \times eR \cong (fR/r.ann(ea) \cap fR) \times eR \cong (e+f)R/r.ann(ea) \cap fR$. This gives by hypothesis, $eafR \times eR$ is π-injective. Since $eafR \subset eR$, and eR is indecomposable, $eafR = eR$. Then $eR \cong fR/r.ann(ea) \cap fR)$ gives $eR \cong fR$. To prove that eR is minimal, let $0 \neq eb \in eR$. If $eb(1-e) \neq 0$, then as before $eR = eb(1-e)R$. This implies $eR = ebR$. On the other hand, if $eb(1-e) = 0$, then $ebe \neq 0$. Since $ebeR \oplus fR = (ebe+f)R$, $ebeR \oplus fR$ is π-injective. Furthermore, $eR \cong fR$ implies $ebeR \oplus fR \cong ebeR \times eR$. Therefore, $ebeR \times eR$ is π-injective, and so as explained earlier $ebeR = eR$. This proves eR is a minimal right ideal. □

Theorem 4.2. (*Jain and Mohamed,* [64]). *A ring R is a right cc-ring if and only if $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right valuation right duo rings with nil radical.*

Proof. Since R a continuous ring $J(R) = Z(R_R)$ and idempotents modulo $J(R)$ can be lifted. Also, $R/J(R)$ is a von-neumann regular ring that has finite uniform dimension (Theorem 2.8). This gives $R/J(R)$ is semisimple artinian which yield that R is semiperfect. In what follows, we shall use that if $A \oplus B$ is direct sum of indecomposable projective right ideals such that for some right ideal $C \subset A$, A/C embeds in B and $A/C \times B$ is continuous, then $C = 0$ and $A \cong B$. Since R is semiperfect, write $R = e_1R \oplus ... \oplus e_nR$ as a direct sum of indecomposable continuous right ideals. We can group all isomorphic indecomposable right ideals as explained above, and write after renumbering if necessary $R = [e_1R] \oplus ... \oplus [e_kR]$ as a direct sum of ideals, where each bracket represents the sum of indecomposable right ideals (possibly only one term in the sum) isomorphic to the term in the bracket. By taking endomorphism rings of R-modules on both sides, we obtain $R = M_{n_1}(e_1Re_1) \oplus ... \oplus M_{n_k}(e_kRe_k)$, where each $M_{n_i}(e_iRe_i)$ represents $n_i \times n_i$ matrix ring over e_iRe_i which is local. It is known that matrix ring of size greater than 1 is continuous if and only if it is self-injective and so $M_{n_i}(e_iRe_i)$ is self-injective if $n_i \geq 2$. Furthermore, it can shown that if $eR \times fR$ is continuous with $Hom_R(eR, fR) \neq 0$, and every cyclic is continuous then $eR \cong fR$ is minimal (see, for example, Lemma 2.5 and Proposition 2.6 [64]). All this gives $M_{n_i}(e_iRe_i)$ is semi-simple artinian for each $n_i \geq 2$. Now we consider the ring direct summand S of R which is a local *cc*-ring. We show S is right duo. Let xS be a right ideal and $a \in S$. If $a \notin J(S)$, then $(1-a)$ is unit and so $(1-a)xS \cong xS$. If $(1-a)xS \subset xS$, we obtain $axS \subset xS$. If $xS \subset (1-a)xS$. Since $(1-a)xS$ is continuous, xS is a direct summand of $(1-a)xS$ which is indecomposable as S is local. Hence $xS = (1-a)xS$, proving that $axS = xS$. If $a \in J(S)$, we can similarly show that $axS \subset xS$. So xS and thus every right ideal of S is two-sided. Finally we show $J(S)$ is nil. This is standard argument of taking a non-nil element $a \in J(S)$ and finding a maximal element P in the family of right ideals that does not contain any power of a. This is a prime ideal. Thus S/P is a prime local *cc*-ring and hence right duo. This gives S/P is a continuous integral domain which is then a division ring. This yields $P = J(S)$, a contradiction because $a \in J(S)$. Thus all elements of $J(S)$ are nil. Hence S is a local right duo ring with nil radical. The converse is straightforward. □

Remark 4.3. It has been mistakenly noted in Osofsky-Smith [97] that Jain-Mohamad theorem was proved for semiperfect rings. The remark toward the end in the paper [64] stated that a *cc*-ring is always semiperfect.

Lemma 4.4. (*Jain and Mueller,* [66]). *Let R be a ring over which each proper cyclic module is continuous. Then*

(a) Eeah nonzero prime ideal is maximal.

(b) R is either prime or semiperfect with nil radical.

Theorem 4.5. (*Jain and Mueller,* [66]). *Let R be a semiperfect ring over which each proper cyclic right module is continuous. Then R is one of the following types:*

(i) $R = \oplus A_i$, where each A_i is a simple artinian right valuation right duo ring with nil radical, or a local ring whose maximal ideal M satisfies $M^2 = 0$ and $l(M) = 2$.

(ii) $R = \begin{pmatrix} \Delta & V \\ 0 & D \end{pmatrix}$ where D and Δ are division rings and V is a one-dimensional right vector space over D.

Proof. Assume that each proper cyclic R-module is continuous. If $R = \oplus_{i=1}^{n} e_iR$ with $n \geq 3$, then we can show that every cyclic module is continuous and structure of this class of rings, called *cc*-rings has already been given in an earlier theorem

(Theorem 4.2). The proof is exactly on the same lines as for $PCQI$-rings. The reader may refer to (Proposition 7, p. 464, [71]).

Case 1. Assume R is local. Let I be a non-zero right ideal of R, then R/I is continuous and hence uniform. We claim that if there exist non-zero right ideals A and B of R such that $A \cap B = 0$ then A, B are minimal right ideals and $S = A \oplus B$ where S is the right socle of R. If X is a non-zero right ideal of R and $X \subset A$ then R/X is uniform. But $A/X \cap B/X = 0$ gives $A = X$. It is immediate now that $S = A \oplus B$. Let M be the unique maximal ideal of R, and let $x \in M$, $x \notin S$. Then xR must be an essential right ideal, for otherwise xR will be minimal. This implies $S \subset xR$, and thus xR cannot be indecomposable. Therefore, $xR = X_1 \oplus X_2$ for some non-zero right ideals X_1, X_2. But then $S = X_1 \oplus X_2$, a contradiction since $x \in S$. Hence $S = M$, which gives $M^2 = 0$ and $l(M) = 2$. Next, if each nonzero right ideal is essential, it follows immediately that R is a right valuation ring. To prove that R is right duo, consider right ideal aR and $x \in R$. Then either $xaR \subset aR$ or $aR \subset xaR$. If $xaR \subset aR$, for all $x \in R$, then aR is two-sided. So assume $aR \subset xaR$ for some $x \in R$. In case $x \notin M$ then $xaR \cong aR$. Since xaR is continuous, we get $xaR = aR$. In case $x \in M$, consider $1 - x$ and proceed as before. Thus aR is a two-sided ideal. Hence R is a right valuation right duo ring.

Case 2. Consider the case $R \cong e_iR \oplus e_jR$. Assume $e_iRe_j \neq 0$. If e_jRe_i is also not zero then e_iR, e_jR are subisomorphic to each other. It follows then $e_iR \cong e_jR$. So every nonzero homomorphism $e_jR \to e_iR$ is a monomorphism. Next, we claim $e_jNe_j = 0$. Else, choose a nonzero element $e_jae_j \in e_jNe_j$. This induces an R-homomorphism $f : e_jR \to e_jR$ given by $f(e_jx) = e_jae_j(e_jx)$. f is not monomorphism because e_jae_j is a nil element. Let $0 \neq g \in Hom_R(e_jR, e_iR)$. Then $0 \neq gf \in Hom_R(e_jR, e_iR)$. But gf is not a monomorphism, a contradiction. Hence $e_jNe_j = 0$ and so e_jRe_j is a division ring. This proves R is simple artinian.

Consider now the case $e_jRe_i = 0$. Then $e_jN = e_jN(e_i + e_j) = e_jNe_j = 0$, and so e_jR is a minimal right ideal. This gives e_jRe_j is a division ring. In this case, since e_iR is uniform, for all $0 \neq e_ixe_j \in e_iRe_j$, e_ixe_jR is the unique minimal right ideal in e_iR. But then $e_ixe_jR = e_iRe_jR$, for all $0 \neq e_ixe_j \in e_iRe_j$. We proceed to prove that $e_iNe_i = 0$ and thus e_iRe_i is also a division ring. If possible, let $0 \neq e_ixe_i \in e_iNe_i$. Consider the mapping $\sigma : e_iR \to e_iR$ given by left multiplication with e_ixe_i. Since e_ixe_i is nil, σ is not one-one. Thus $ker(\sigma)$ contains the uniquie minimal right ideal e_iRe_jR. Therefore, $e_ixe_iRe_jR = 0$. But then $e_ixe_iRe_j = 0$. Now the unique minimal right ideal e_iRe_jR is contained in every nonzero right ideal. So $e_iRe_jR \subset e_ixe_iR$. Therefore, $e_iRe_jRe_j \subset e_ixe_iRe_j = 0$. Since e_jRe_j is a division ring, we obtain $e_iRe_j = 0$, a contradiction. Thus $e_iNe_i = 0$, which gives e_iRe_i is a division ring.

We now prove that e_iRe_j is a one-dimensional right vector space over e_jRe_j. Let N denote the radical of R. Then $e_iN = e_iNe_j$ because $e_iNe_i = 0$. Since e_iRe_jR is a unique minimal right ideal in e_iR, $e_iN \supseteq e_iRe_jR$. Because $e_jRe_i = 0, e_iRe_jR = e_iRe_jRe_j$. This implies $e_iRe_jR = e_iRe_j$ because e_jRe_j is a division ring. Furthermore, e_iRe_j is a right ideal as shown above and $(e_iRe_j)^2 = 0$. This implies $e_iRe_j \subset e_iN$, and so $e_iRe_j = e_iN = e_iRe_jR = e_ixe_jRe_j$. This proves e_iRe_j is a one-dimensional right vector space over e_jRe_j. Hence, $R = \begin{pmatrix} e_iRe_i & e_iRe_j \\ 0 & e_jRe_j \end{pmatrix} \cong \begin{pmatrix} \Delta & V \\ 0 & D \end{pmatrix}$ where D and Δ are division rings and V is a one-dimensional right vector space over e_jRe_j. □

5. Rings whose Cyclic or Proper Cyclic Modules are Quasi-continuous (π-Injective)

The class of right *cc*-rings was further generalized by Goel and Jain [33] who studied rings over which each cyclic module is quasi-continuous (in other words, each cyclic module is π-injective). Goel and Jain called such rings πc-rings and obtained the following results.

Theorem 5.1. (*Goel and Jain,* [34]).

(i) Let R be a right self-injective ring. Then R is a right πc-ring if and only if $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right self-injective right valuation rings.

(ii) Let R be a semiperfect ring. Then R is a right πc-ring if and only if $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right valuation rings.

(iii) Let R be a right πc-ring with zero right singular ideal. Then $R = A \oplus B$ where A is semisimple artinian and B is a finite direct sum of right Ore domains.

By appealing to the Theorem 2.8, we can prove, in general, the following theorem whence the above theorem comes as special case.

Theorem 5.2. *If R is a right πc-ring, then $R = R_1 \oplus R_2 \oplus .... \oplus R_k$, where R_i is either simple artinian or a right uniform ring.*

Proof. By Theorem 2.8, R is a direct sum of uniform right ideals. Write $R = e_1R \oplus ... \oplus e_kR$ as a direct sum of uniform π-injective (= quasi-continuous) right ideals. It can be shown that if $e_iR \times e_jR$ is π-injective such that $Hom(e_iR, e_jR) \neq 0$ then $e_iR \cong e_jR$ minimal right ideals (see [33], Corrolary 1.13 and Lemma 2.3). By summing all isomorphic right ideals e_iR we can show that the sum is indeed two-sided ideal. This proves the desired result. □

We now consider semiperfect rings over which each proper cyclic right module is π-injective (= quasi-continuous).

We call a ring R to be a right $pc\pi i$-ring if each proper cyclic right R-module is π-injective. The following lemma follows from the definition of π-injective module.

Lemma 5.3. *Let A and B be R-modulcs such that A is embeddable in B. If $A \times B$ is π-injective then the exact sequence $0 \to A \to B$ splits.*

If A, B are continuous R-modules such that A is embeddable in B and B is embeddable in A, then it is well-known that $A \cong B$. However, this is not true for π-injective modules, in general. But, if both A and B are indecomposable projective and $A \times B$ is π-injective, then $A \cong B$.

Lemma 5.4. *If M_1 and M_2 are π-injective such that $M_1 \times M_2$ is π-injective and $E(M_1) \cong E(M_2)$ then $M_1 \cong M_2$.*

Theorem 5.5. *Let $R = \oplus_{i\in I} e_i R$ be a semiperfect ring with nil radical, where $e_i R$ are indecomposable right ideals and $|I| \geq 3$. Then R is right a pcπi-ring if and only if $R = \oplus R_i$ where each R_i is either simple artinian or a right valuation ring.*

Proof. Since the number of summands is more than 2, for each pair of indecomposable right ideals e_iR, e_jR, we have $e_iR \times e_jR$ is π-injective. Suppose $e_iRe_j \neq 0$. By above lemmas, $e_iR \cong e_jR$ since $e_iR \times e_jR$ is π-injective and the Lemma 4.1 applies to give e_iR is minimal. If $e_iRe_j = 0$ and $e_jRe_i = 0$ for all $j \neq i$ then e_iR is a two-sided ideal and is a ring direct summand of R which is a right valuation ring. This proves the theorem. □

Lemma 5.6. *Let $R = e_1R \oplus e_2R$ be a semiperfect ring with nil radical N, where e_1R, e_2R are indecomposable right ideals such that $e_1Re_2 \neq 0$ and $e_2Re_1 \neq 0$. Then R is a right pcπi-ring if and only if it is simple artinian.*

Proof. Suppose R is a right *pcπi*-ring. By hypothesis, there exists a nonzero homomorphism $f : e_1R \to e_2R$. This gives $R/\ker f \cong e_1R/\ker f \times e_2R$. If $R/\ker f \cong R$ then $R \cong R \times \ker f$, and this implies $\ker f = 0$, because R is semiperfect. If $R/\ker f \ncong R$, then $e_1R/\ker f \times e_2R$ is π-injective and $e_1R/\ker f$ embeds in e_2R. This implies $e_1R/\ker f \cong e_2R$ and so $ker f = 0$. In any case, e_1R is embeddable in e_2R and similarly e_2R is embeddable in e_1R. This yields either $e_1R \cong e_2R$, or e_1R embeds in e_2N and e_2R embeds in e_1N. Therefore, either $e_1R \cong e_2R$ or R embeds in N. The latter is impossible. Hence $e_1R \cong e_2R$.

We now show $e_1Ne_1 = 0$. Since$N$ is nil, there exists (e_1xe_1) such that $(e_1xe_1)^{k-1} \neq 0$ but $(e_1xe_1)^k = 0$. Consider the mapping $g : e_1R \to e_1R$ given by left multiplication with e_1x. Then the image of $(e_1xe_1)^{k-1}$ is zero, a contradiction because this mapping naturally induces a mapping from e_1R to e_2R which as proved above must be one-to-one. Hence $e_1Ne_1 = 0$. This yields R is a 2×2 matrix ring over a division ring e_1Re_1. □

Theorem 5.7. *Let $R = e_1R \oplus e_2R$ be a semiperfect right pcπi-ring with nil radical N, where e_1R, e_2R are indecomposable right ideals such that $e_1Re_2 \neq 0$ and $e_2Re_1 = 0$. Then e_2R is a minimal right ideal and $ann_{e_1Re_1}(e_1Re_2) = e_1Ne_1$.*

Proof. As in the previous theorem any nonzero map from e_2R to e_1R is a monomorphism. We show $e_2N = 0$. Choose $e_1xe_2 \neq 0$ and if possible let $e_2y \neq 0$. Since N is nil, we can assume $(e_2y)^2 = 0$. Then, $(e_1xe_2ye_2)e_2y = 0$. This yields $e_1xe_2ye_2 = 0$. Again $e_2ye_2 = 0$. This gives $e_2y = 0$ since $e_2Re_1 = 0$, a contradiction. Hence $e_2N = 0$, and so e_2R is unique minimal right ideal in e_2R because e_2R is uniform. The last part is straightforward. For more details the reader may refer to [34]. □

Lemma 5.8. *Under the hypothesis of the above lemma* $A = \begin{bmatrix} e_1Ne_1 & 0 \\ 0 & 0 \end{bmatrix}$ *is an ideal in* $S = \begin{bmatrix} e_1Re_1 & e_1Re_2 \\ & e_2Re_2 \end{bmatrix} \cong R$ *and*

$S/A \cong \begin{bmatrix} e_1Re_1/e_1Ne_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{bmatrix}$ *is not* π*-injective as* S *or* S/A*-module and therefore* R *is not a right pc*π*i-ring if* $A \neq 0$.

Proof. It follows from above lemma that e_1Re_2 is a one-dimensional right vector space over e_2Re_2. Thus we may write $\begin{bmatrix} e_1Re_1/e_1Ne_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{bmatrix} = \begin{bmatrix} K & D \\ 0 & D \end{bmatrix} \cong \begin{bmatrix} K & D \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ as S/A-modules, where K and D are division rings. Since the second summand can be embedded in the first but the embedding is not onto, S/A cannot be π-injective. This completes the proof. □

Theorem 5.9. *Let* $R = e_1R \oplus e_2R$ *be a semiperfect ring with nil radical such that* e_1, e_2 *are primitive orthogonal idempotents,* $e_1Re_2 \neq 0, e_2Re_1 = 0$. *Then* R *is a right pc*π*i-ring if and only if* $R \cong \begin{bmatrix} K & V \\ 0 & D \end{bmatrix}$ *where* K *and* D *are division rings and* V *is one-dimensional right space over* D.

Proof. By Theorem 5.7, e_2R is a minimal right ideal and hence e_2Re_2 is a division ring. Since each proper ring homomorphic image of R is a πc-ring it follows that $e_1Ne_1 = 0$. Thus $R \cong \begin{bmatrix} K & V \\ 0 & D \end{bmatrix}$ where K and D are division rings and V is one-dimensional right space over D. Thus the "only if" part of the theorem is completed. The converse is straightforward. □

To complete our discussion we need to give characterization of local *pc*π*i*-rings, This is contained in the following theorem whose proof is similar to the proof of local rings over which every proper cyclic module is continuous. However, note that the local right *pc*π*i*-ring need not be right duo.

Theorem 5.10. *Let* R *be a local ring with unique maximal ideal* N. *Then* R *is a right pc*π*i-ring if and only if* R *is either a right valuation ring or* N^2*=0, and the composition length of* N *is* 2.

In summary, we have obtained the following description of non-local right *pc*π*i*-rings.

Theorem 5.11. *(a) Let* R *be a non-local semiperfect right pc*π*i-ring Then either* R *is a direct sum of rings of the following types (not necessarily all)*

(i) semisimple artinian ring.

(ii) right valuation rings with maximal ideal M *satisfying* $M^2 = 0$*,and* $l(M)=$ 2.

(iii) $\begin{pmatrix} K & V \\ 0 & D \end{pmatrix}$ *where K and D are division rings and V is one-dimensional right space over D.*

Gomez-Pardo and Guil Asensio studied rings over which a family of modules that are quasi-continuous with respect to pure-essential sequences [38]. A right module M is called a *pqc*-module when (i) every pure submodule of M is purely essential in a direct summand of M, and (ii) if U, V are direct summands of M such that $U \cap V = 0$ and $U \oplus V$ is pure in M, then $U \oplus V$ is a direct summand of M. A module M will be called a completely *pqc*-module when every pure quotient of M is a *pqc*-module. The main result in (Theorem 2.9, [38]) shows that if M is a finitely presented completely *pqc*-module, then M has ACC (and DCC) on direct summands and so it is a (finite) direct sum of indecomposables.

It will be interesting to study rings over which every proper quotient of M is a finitely presented (i) completely *pqc*-module, or (ii) a completely pure injective module. For definitions of purely essential and other relevant terms, the reader may refer to [38].

6. Rings over which Cyclic Modules are Weakly Injective

In [61] Jain, Lopez-Permouth and Singh studied rings whose each cyclic module is weakly-injective and proved the following.

Theorem 6.1. (*Jain, Lopez-Permouth and Singh,* [61]). *The following conditions on a ring R are equivalent:*

(a) R is right weakly-semisimple, that is, each right R-module is weakly-injective.

(b) Each cyclic right R-module is weakly R^2-injective and R is right noetherian.

(c) Each cyclic uniform right R-module is weakly R^2-injective and R is right noetherian.

Proof. $(a) \implies (b)$. Let R be a right weakly-semisimple ring. We will show that R must be right noetherian. Let M be a quasi-injective right R-module. By assumption, M is weakly-injective. Suppose M is not injective. Then by Zorn's lemma, there exists a submodule A of $E(M)$ and a homomorphism $f : A \to M$ which cannot be extended to any $f' : B \to M$ with B a submodule of $E(M)$ containing A properly. Let $b \in E(M)/A$. As $A \subset_e E(M)$, we have $C = bR \cap A \neq 0$. Let $f_1 : C \to M$ be the restriction of f to C. As M is weakly-injective, bR embeds in M. Therefore, M is bR-injective and f_1 extends to $g : bR \to M$. Define $f' : A + bR \to M$ by $f'(a + br) = f(a) + g(br)$ for $a \in A, r \in R$. Since f' extends f, this yields a contradiction. Hence, M must be injective. Thus, R is a right QI ring and hence by Boyle, R must be right noetherian.

Clearly $(b) \implies (c)$.

$(c) \implies (a)$. Suppose each cyclic right R-module is weakly R^2-injective and R is right noetherian. Let M be any right R-module and $x_1, ..., x_n \in E(M)$. Let $K = \Sigma_{i=1}^n x_i R$. It follows that $E(M) = E(K) \oplus L$ for some submodule L of $E(M)$ and $E(K)$ has finite uniform dimension. Therefore, $M \cap E(K)$, being essential in $E(K)$, has finite uniform dimension. Let N be a finite direct sum of uniform cyclic modules which is essential in $M \cap E(K)$. Note that if a cyclic right R-module is weakly R^2-injective then it is weakly-injective. Since over a right $q.f.d.$ ring every direct sum of weakly-injective modules is again weakly-injective, it follows that N is weakly-injective. Thus there exists an automorphism $\sigma : E(K) \to E(K)$ with $K \subseteq \sigma(N) \subseteq \sigma(M \cap E(K))$. Clearly σ extends to an automorphism ϕ of $E(M)$ which is identity on L. This automorphism satisfies $K \subseteq \phi(M)$. Hence M is weakly-injective. Thus, every right R-module is weakly-injective. Therefore, R is right weakly-semisimple. □

7. Rings over which cyclic modules are quasi-projective

Rings over which each cyclic module is projective are obviously semisimple artinian. The study of rings over which each cyclic module is quasi-projective was initiated by Koehler. She called a ring a right q^*-ring if each cyclic module is quasi-projective. Koehler proved the following.

Theorem 7.1. (*Koehler,* [75]).

(i) A semiperfect ring R is a left q^*-ring if and only if every left ideal in the Jacobson radical of R is an ideal.

(ii) Let R be a left self-injective semiperfect left q^*-ring. Then $R = A \oplus B$, where A is semisimple artinian and B is a finite direct sum of paiwise non-isomorphic uniform left ideals.

(iii) If R is a prime semiperfect left q^*-ring then R is either semisimple artinian or R is local.

(iv) Let R be a quasi-Frobenius ring. Then R is a left q^*-ring if and only if R is a right q^*-ring.

(v) A proper matrix ring $\mathbb{M}_n(R)$, $(n > 1)$ is a left q^*-ring if and only if R is semisimple artinian.

Proof. (i) To prove this we first note the following useful facts proved by Miyashita [84] and Wu-Jans [106]. Let P be a projective module, $\phi : P \to M$ be an endomorphism and $S = End(P)$. Then

(a) M is quasi-projective if $ker(\phi)$ is invariant under S, and

(b) $ker(\phi)$ is invariant under S if $ker(\phi)$ is small in P and M is quasi-projective.

Let R be a semiperfect left q^*-ring. Let I be a left ideal of R contained in $J(R)$. Consider $\phi : R \to R/I$. Since I is small in R and R/I is quasi-projective, I is invariant under $End(R) \cong R$ by (b) above. Hence I is an ideal. Conversely, assume that every left ideal in $J(R)$ is an ideal. Let K be a left ideal in R. Then R/K has a projective cover $\phi : P \to R/K$, and P can be considered to be a direct summand of R. Since $ker(\phi)$ is small in P, it is small in R and contained in $J(R)$. If $f \in End(P)$, then there is an $r \in R$ such that $f(x) = xr$ for every $x \in P$. Therefore, $ker(\phi)$ is invariant under $End(P)$. Hence by (b) above, R/I is quasi-projective. Therefore, R is a left q^*-ring.

(ii) Since R is semiperfect, $R = Re_1 + ... + Re_k + Re_{k+1} + ... + Re_n$, where $e_1, ..., e_n$ are orthogonal indecomposable idempotents. We may assume $Re_1, ..., Re_k$ are all the simple components of the decomposition. By (i), $J(R)e_i.e_iRe_j = 0$ if $i \neq j$. Since $Hom(Re_i, Re_j) = e_iRe_j$, $Re_i \ncong Re_j$, for $i, j > k$ and $i \neq j$. Let $A = Re_1 + ... + Re_k$, and $B = Re_{k+1} + ... + Re_n$. To complete the proof we show that A and B are ideals. Let $i \leq k$ and $j > k$. Then $Re_i.e_iRe_j$ is 0 or simple and is contained in Re_j. Since Re_i is a simple injective module, and Re_j is not simple and is indecomposable, $Re_i.e_iRe_j = 0$. Similarly $Re_j.e_jRe_i = 0$ because Re_i is a simple projective module.

(iii) Since R is semiperfect, $R = Re_1 + ... + Re_n$ where $e_1, ..., e_n$ are nonzero orthogonal indecomposable idempotents. If $n = 1$, then R is local because $J(R)e_1$ is a unique maximal left ideal in Re_1. From (i), $J(R)e_i.J(R)e_j \subseteq J(R)e_i \cap J(R)e_j = 0$ for $i \neq j$. Hence $J(R)e_i = 0$ for all but at most one i, say $i = k$, because R is prime. Since $e_kRe_i \neq 0$ (because R is prime), there is an epimorphism from Re_k to Re_i. So, $Re_k \cong Re_i$ for $i = 1, 2, ..., n$ because Re_i is simple and projective, and Re_k is indecomposable. Therefore R is simple artinian if $n > 1$.

(iv) Let R be a quasi-Frobenius left q^*-ring. We first claim that each essential left ideal of R is an ideal. Let I be an essential left ideal of R. Then R/I has a projective cover $\phi : P \to R/I$. Let $f : R \to R/I$ be the canonical homomorphism. Then there is an epimorphism $f' : R \to P$ such that $\phi \circ f' = f$ because P is projective and $ker(\phi)$ is small. Since P is projective and f' is onto, $R = Re_1 \oplus Re_2$ with $Re_1 \cong P$. Also, it can be seen that $I = K \oplus Re_2$ where $K \cong ker(\phi)$ and $K \subseteq Re_1$. The left ideal K is small in R, and R is a left q^*-ring. Thus $K \subseteq J(R)$, and K must be an ideal by (i). Now as I is an essential left ideal of R, the left socle $Socle_l(R)$ of R is contained in I. The left ideal $J(R)e_2$ is an ideal in R. So $Re_2.e_eRe_1 \subseteq Socle_l(R)$ because $J(R)e_2.e_2Re_1 = 0$ and R is semiperfect. Therefore, I is an ideal. Next, we claim that each left ideal of R is quasi-injective. Let A be any left ideal of R. If B is a complement of A in R then $A \oplus B$ is an essential left ideal of R. We have just now shown that $A \oplus B$ must be an ideal of R. Since R is right self-injective, $A \oplus B$ is quasi-injective and hence A is quasi-injective. Thus each left ideal of R is quasi-injective. Now, we proceed to show that R must be right q^*. Let L be a right ideal contained in $J(R)$ and $a \in L$. Since R is left self-injective, $J(R) = \{a : ann_l(a) \text{ is an essential left ideal of } R\}$ and $ann_r(ann_l(aR)) = aR$. In addition, $ann_l(aR) = ann_l(a)$ which is an essential left ideal of R. Therefore, aR is an ideal, that is, L is an ideal. Hence, by (i), R is a right q^*-ring. The converse is similar.

(v) For proof of this part, the reader is referred to the Theorem 2.5, [75]. □

Koehler gave example of a ring which is both left and right artinian, a right q^*-ring but not a left q^*-ring.

Example. (Koehler, [75]). Let R be the ring of matrices of the form $\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix}$ such that $\bar{a} \in \mathbb{Z}_4$ and $\bar{b}, \bar{c} \in \mathbb{Z}_2$. Clearly, $J(R)$ consists precisely of the matrices of the form $\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{0} \end{bmatrix}$ such that $\bar{a} = \bar{0}$ or $\bar{2} \in \mathbb{Z}_4$ and $\bar{b} \in \mathbb{Z}_2$. Every right ideal in $J(R)$ is an ideal. However the left ideal I consisting of exactly two elements $\begin{bmatrix} \bar{0} & \bar{0} \\ 0 & \bar{0} \end{bmatrix}$ and $\begin{bmatrix} \bar{2} & \bar{1} \\ 0 & \bar{0} \end{bmatrix}$ is not an ideal. Therefore, by (i) of the above theorem, R is a right q^*-ring but not a left q^*-ring.

8. Hypercyclic, q-Hypercyclic and π-Hypercyclic Rings

Caldwell [15] initiated study of rings whose each cyclic module has a cyclic injective hull. Such rings are called *hypercyclic rings.* A ring R is called restricted hypercyclic if R is hypercyclic and $R/J(R)$ is artinian. Caldwell proved the following.

Theorem 8.1. (*Caldwell,* [15]). *A commutative hypercyclic ring must be restricted.*

Proof. Let R be a commutative hypercyclic ring. Then R is self-injective. Therefore, $R/J(R)$ is a self-injective von Neumann regular ring. Clearly, $R/J(R)$ is either semisimple artinian or it has an infinite set of orthogonal idempotents. Suppose $R/J(R)$ has an infinite set of orthogonal idempotents. Since orthogonal idempotents can be lifted orthogonally modulo $J(R)$, R has an infinite set of orthogonal idempotents, say $\{e_i\}$. Let $I = \Sigma e_i R$. Then R/I would be non-injective, a contradiction to the fact that if $\{e_i\}$ is any set of idempotents in a commutative hypercyclic ring R, and if $I = \Sigma e_i R$ then R/I is an injective R-module. □

Osofsky continued the study of hypercyclic rings and showed the following.

Theorem 8.2. (*Osofsky,* [94]). *A ring is restricted hypercyclic if and only if it is a ring direct sum of matrix rings over hypercyclic local rings.*

Rosenberg and Zelinsky [99] considered rings over which each cyclic module has injective hull of finite length. This led Jain and Saleh [67] to consider rings over which each cyclic module has finitely generated injective hull. A ring R is called q-hypercyclic if each cyclic ring R-module has a cyclic quasi-injective hull.

Lemma 8.3. (*Jain and Saleh,* [67]).

(a) Let R be any ring such that $R/J(R)$ is semisimple artinian and $J \neq J^2$. Then any simple R-module A that is not injective can be embedded in J/J^2.

(b) Let M be an injective R-module and I, K be ideals in R, where $K \subseteq I$. Then $Hom(I/K, M) \cong ann_M(K)/ann_M(I)$ as R-modules.

(c) Let R be a ring such that every cyclic R-module has finitely generated quasi-injective hull. Then every ring homomorphic image of R has this property.

Proof. (a) This is straightforward. (b) The proof follows from the canonical embedding of $ann_M(K)/ann_M(I)$ into $Hom(I/K, M)$ and the Baer criterion for injective modules. (c) Let A be a two-sided ideal of R. Let $\bar{R} = R/A$ and $\bar{R}/\bar{I}$ be a cyclic $\bar{R}$-module, where $\bar{I} = I/A$. We have $\bar{R}/\bar{I} \cong R/I$ as R-modules. Denote by P the quasi-injective hull of R/I as an R-module. P is a finitely generated R-module. Since $P = End_R(E(R/I))R/I$, A annihilates P and hence P is an R/A-module and indeed P is quasi-injective as an R/A-module. If $B \subset P$, and B is quasi-injective as an R/A-module, then B is also quasi-injective as an R-module. Hence, P is the quasi-injective hull of R/I as an R/A module. Since P is a finitely generated R-module and A annihilates P, P is also finitely generated as an R/A-module. □

Theorem 8.4. (*Jain and Saleh,* [67]). *Let R be a right artinian ring. The following statements are equivalent.*

(a) Every cyclic R-module can be embedded in a finitely generated injective module.

(b) $Hom(J/J^2, A)$ is finitely generated for every simple R-module A.

(c) Every cyclic R-module has finitely generated quasi-injective hull.

(d) The injective hull of R/J^2 is a finitely generated R-module .

Proof. $(a) \implies (c)$: Let M be a cyclic R-module. Since $q.inj.hull(M) \subset E(M)$, and R is artinian, it follows that $q.inj.hull(M)$ is also finitely generated. $(c) \implies (d)$: Let E_1 and E_2 denote respectively the injective hull of R/J as R/J-module and as R-module. By Lemma 8.3, the quasi-injective hull of every cyclic R/J^2-module is finitely generated. In particular, $q.inj.hull(R/J^2)$ $(= E_1)$ is a finitely generated R/J^2-module. This implies $Hom(J/J^2, I/J^2)$ is also finitely R/J^2-module generated for any minimal right ideal I/J^2 of R/J^2 by Rosenberg-Zelinsky [99]. But then $Hom(J/J^2, I/J^2)$ is a finitely generated K-module, where $K = (R/J^2)/(J/J^2) \cong R/J$. Therefore, $Hom_{R/J^2}(J/J^2, I/J^2)$ is a finitely generated R/J-module, and so $Hom_R(J/J^2, I/J^2)$ is a finitely generated R-module for each simple submodule I/J^2 of R/J^2. This yields E_2, the injective hull of R/J^2 as an R-module, is finitely generated (see [99], Theorem 1). $(d) \implies (b)$: Let A be a simple R-module. If A is injective, then by the above lemma, $Hom(J/J^2, A) = 0$, or A which is finitely generated. If A is not injective, then by above lemma, A can be embedded in J/J^2. So there exists a right ideal I of R such that $I \subset J$ and $A = I/J^2$. Since I/J^2 is a simple submodule of R/J^2, and the injective hull of R/J^2 as an R-module is finitely generated, $Hom_R(J/J^2, I/J^2)$ is finitely generated. Therefore, $Hom_R(J/J^2, A)$ is finitely generated. It remains to see that $(b) \implies (a)$, and this holds by ([99], Theorem 1) and the above lemma. □

Next, we consider rings with Krull dimension over which each cyclic module has cyclic injective hull. Indeed such rings turn out to be artinian rings. We shall need the following.

Lemma 8.5. (*Gordon and Robson,* [39]). *If R is a ring with Krull dimension $K(R)$ then $K(R) = K(R/P)$, for some prime ideal P. In fact P is a minimal prime ideal.*

The following is straightforward.

Lemma 8.6. *If R is a valuation ring, and $P = aR$ is a nonzero prime ideal $P \neq J$, then R/P is not a domain.*

Proposition 8.7. (*Jain and Saleh,* [67]). *Let R be a local hypercyclic ring. If R has Krull dimension, then R is right artinian.*

Proof. First suppose $J = J(R)$ is nil. Then, since R has Krull dimension, J is nilpotent. This implies J is the only prime ideal of R. But then by Lemma 8.5, $K(R) = K(R/J) = 0$, which proves that R is artinian. Suppose J is not nil. Then by Osofsky ([94], Theorem 2.12) there exists a nonzero nilpotent ideal $aR \subset J$, $a \in R$ such that aR is the maximal proper two-sided ideal below J. Since R has Krull dimension, R satisfies *acc* on prime ideals. If J is not the only prime ideal, there exists a prime ideal Q such that Q is maximal among all the prime ideals different from J. Then $Q \subset aR$. Since aR is nilpotent, $Q = aR$. Thus $Q \neq (0)$, since aR is not zero. So by Lemma 8.6 and R being valuation, Q is not completely prime. Consider now the prime ring R/Q. Since R/Q has Krull dimension, it is a prime Goldie ring. Therefore $Z(R/Q)$, the right singular ideal of R/Q, is zero. Thus $ann(\bar{x}) = \bar{0}$, for every $\bar{0} \neq \bar{x} \in R/Q$, since R is a valuation ring. Hence Q is completely prime, which is a contradiction. This proves that J is the only prime ideal. Therefore, $K(R) = K(R/J) = 0$, and hence R is artinian. □

Theorem 8.8. *Let R be a hypercyclic ring. Then R has Krull dimension if and only if R is artinian.*

Proof. If R has Krull dimension, then each homomorphic image of R has *acc* on direct summands. Thus by Osofsky ([94], Lemma 1.7), R has *acc* on direct summands. Thus, by Osofsky, R is a ring direct sum of matrix lings over local hypercyclic rings and hence R is artinian. □

Corollary 8.9. (*Jain and Saleh,* [67]). *A hypercyclic ring with Krull dimension is quasi-Frobenius.*

Dinh, Guil Asensio and Lopez-Permouth studied hereditary ring R such that the injective hull $E(R_R)$ is finitely generated (or cyclic) and proved the following which answers in the affirmative a question posed by Dauns in [26] and Gomez-Perdo, Dung and Wisbauer in [36].

Theorem 8.10. (*Dinh, Guil Asensio, Lopez-Permouth,* [27]). *Let R be a right hereditary ring. If the injective hull $E(R_R)$ is finitely generated (or cyclic), then R is right artinian.*

More generally, they obtained the following.

Theorem 8.11. (*Dinh, Guil Asensio, Lopez-Permouth,* [27]). *Let R be a right hereditary ring. If the injective hull $E(R_R)$ is countably generated, then R is right noetherian.*

For a commutative ring R, R can be shown to be q-hypercyclic (= qc-ring) if R is hypercyclic. Whether a hypercyclic ring (not necessarily commutative) is q-hypercyclic is considered by showing that a local hypercyclic ring R is q-hypercyclic if and only if the Jacobson radical of R is nil. However, It is not known if there exists a local hypercyclic ring with non-nil radical.

Lemma 8.12. *Let R be semiperfect and q-hypercyclic. Then R is right self-injective.*

Proof. Let I be a right ideal of R such that R/I is the quasi-injective hull of R. Let $\phi : R \to R/I$ be the embedding. Since R/I contains a copy of R, R/I is injective. Let $\phi(R) = B/I$. Then $B/I \subset_e R/I$. Hence $B \subset_e R$. Since $R \cong B/I$, B/I is projective. Thus $B = I \oplus K$ for some $K_R \subseteq B_R$. Now $R \cong B/I = (I \oplus K)/I \cong K$. Therefore $E(R) \cong E(K)$. But then $I \oplus K \subseteq_e R$ implies $E(R) = E(l) \oplus E(K) \cong E(l) \oplus E(R)$. Since $E(R) \cong R/I$, $E(R)$ is a finite direct sum of indecomposable modules, by [94]. Thus $E(R)$ has finite Azumaya-Diagram. Therefore, $E(R) \cong E(R) \oplus E(l)$ implies $E(I) = 0$. Hence $I = 0$. Thus R is right self-injective. □

Lemma 8.13. (*Jain and Malik,* [63]). *Let R be q-hypercyclic. Then every homomorphic image of R is also q-hypercyclic.*

Proof. Let A be a two-sided ideal of R. Let $\bar{R} = R/A$. Let $\bar{R}/\bar{I}$ be a cyclic $\bar{R}$-module, where $\bar{I} = I/A$. But $\bar{R}/\bar{I} \cong R/I$. Since $A \subset I$, A annihilates R/I. Let R/K be the quasi-injective hull of R/I as an R-module. Then $R/K \cong End_R(E(R/I))R/I$. Then it follows that A annihilates R/K. Thus R/K may be regarded as an $\bar{R}$-module. Since R/K is quasi-injective as an R-module, R/K is quasi-injective as an R-module. Since A is a two-sided ideal and annihilates R/K, $A \subset K$. Hence $R/K \cong (R/A)/(K/A)$. Clearly $\bar{R}/\bar{K}$ is the quasi-injective hull of $\bar{R}/\bar{I}$ as an $\bar{R}$-module. Hence $\bar{R}$ is q-hypercyclic. □

Jain and Malik [63] studied local q-hypercyclic rings and obtained the following results.

Lemma 8.14. (*Jain and Malik,* [63]). *Let R be a local q-hypercyclic ring. Then*
(i) Both right and left ideals of R are linearly ordered.
(ii) R is left bounded or right bounded.
(iii) R is a duo ring.

Theorem 8.15. (*Jain and Malik,* [63]).
(i) Let R be a local ring. Then R is q-hypercyclic if and only if R is a qc-ring.
(ii) Let R be a local hypercyclic ring. Then R is q-hypercyclic if and only if $J(R)$ is nil.

Proof. (i) Let R be q-hypercyclic and let A be a non-zero right ideal of R. Then by above lemma, A is a two-sided ideal of R. But then by Lemma 8.13, R/A is a self-injective ring. Thus R/A is a quasi-injective R-module, proving that R is a qc-ring. The converse is obvious.

(ii) Let R be a local hypercyclic ring with $J(R)$ nil. Then by [94], R is a duo, self-injective, valuation ring. But then R is maximal. Therefore, R is a qc-ring and hence a q-hypercyclic ring. Conversely, suppose R is a local q-hypercyclic ring. Let $a \in J(R)$. Suppose $a^n \neq 0$ for any positive integer n. Let $S = \{a^n : n > 0\}$. By Zorn's lemma there exists an ideal P of R maximal with respect to the property that $P \cap S = \emptyset$. Then P is prime. Hence R/P is a prime local q-hypercyclic ring. Thus R/P is either left bounded or right bounded. Then it follows that R/P is a domain. Since R/P is also local and q-hypercyclic, R/P is self-injective and hence a division ring. Therefore P is a maximal ideal of R. Thus $P = J(R)$, a contradiction. Hence $J(R)$ is nil. □

Theorem 8.16. (*Jain and Malik,* [63]).

(i)Let R be a commutative q-hypercyclic ring. Then R must be self-injective.

(ii) Let R be a commutative ring. Then R is q-hypercyclic if and only if R is a qc-ring.

(iii) Let R be a commutative hypercyclic ring. Then R is q-hypercyclic

Proof. (i) This is obvious.

(ii) This is similar to the proof of Theorem 8.15 (i).

(iii) Let R be a commutative hypercyclic ring. Then by ([15], Theorem 2.5), R is a finite direct sum of commutative local hypercyclic rings. So it suffices to show that a commutative local hypercyclic ring is q-hypercyclic. Let R be commutative local and hypercyclic. Then by [15], R is valuation and self-injective, and $J(R)$ is nil. Then by ([74], Theorem 2.3), R is maximal. Since $J(R)$ is nil, R has rank 0. Then R is rank 0 maximal valuation ring. Thus R is a qc-ring [76], proving the theorem. □

The following example of Jain and Malik shows that a q-hypercyclic ring need not be hypercyclic.

Example. (Jain and Malik, [63]). Let F be a field, and x be an indeterminate over F. Let $W = \{\{\alpha_i\} : \{\alpha_i\}$ is a well ordered sequence of nonnegative real numbers$\}$. Let $T = \{\Sigma_{i=0}^{\infty} a_i x^{\alpha_i} : a_i \in F, \{\alpha_i\} \in W\}$ and $R = \frac{T}{xJ(T)}$. Let S be the socle of R. Then R/S is a q-hypercyclic ring but not a hypercyclic ring.

Jain and Malik also obtained an anologue of Osofsky's result by considering semiperfect q-hypercyclic ring.

Theorem 8.17. (*Jain and Malik,* [63]). *Let R be a semiperfect q-hypercyclic ring. Then R is a finite direct sum of q-hypercyclic matrix rings over local rings.*

Proof. $R = e_1R \oplus ... \oplus e_nR$, where e_i, $1 \leq i \leq n$ are primitive idempotents. We will show that for $i \neq j$, either $e_iR \cong e_jR$, or $Hom_R(e_iR, e_jR) = 0$. Suppose for some $i \neq j$, $Hom_R(e_iR, e_jR) \neq 0$. By renumbering, if necessary, we may assume that $i = 1, j = 2$. Let $\alpha : e_1R \rightarrow e_2R$ be a non-zero R-homomorphism. Then

$e_1R/Ker(\alpha)$ embeds in e_2R. Since e_2R is indecomposable, $E(e_1R/Ker(\alpha)) \cong e_2R$. Hence $B = e_2R \oplus ... \oplus e_nR$ contains a copy of $E(e_1R/Ker(\alpha))$. Now $R/Ker(\alpha) \cong (e_1R)/Ker(\alpha) \times e_2R \times ... \times e_nR$. Let $A = (e_1R)/Ker(\alpha)$. Then B is injective and contains a copy of $E(A)$. Hence $Hom_R(B, E(A))B = E(A)$. Since R is q-hypercyclic, for some right ideal I, $R/I \cong q.i.h.(R/Ker(\alpha)) \cong q.i.h.(A \times B) \cong E(A) \times B$. Thus $R/I \cong e_2R \times B$. Then R/I is projective. Hence $R = I \oplus K$ for some right ideal K. Then $K \cong R/I \cong e_2R \times e_2R \times ... \times e_nR$. Thus $R = I \oplus K \cong I \times e_2R \times e_2R \times ... \times e_nR$. Hence by Azumaya Diagram, $e_1R \cong I \times e_2R$. Since e_1R is indecomposable, $I = 0$. Consequently, $R = K$. Then $e_1R \times e_2R \times ... \times e_nR \cong e_2R \times e_2R \times ... \times e_nR$. Again by Azumaya Diagram, $e_1R \cong e_2R$. Thus for $i \neq j$, either $e_iR \cong e_jR$ or $Hom_R(e_iR, e_jR) = 0$. Set $[e_kR] = \Sigma e_iR$, $e_iR \cong e_kR$. Renumbering, if necessary, we may write $R = [e_1R] \oplus ... \oplus [e_tR]$, $t \leq n$. Then for all $1 \leq k \leq t$, $[e_kR]$ is an ideal. Since for any k, $1 \leq k \leq n$, e_kR is indecomposable, $End_R(e_kR) \cong e_kRe_k$ is a local ring. Thus $[e_kR] = \oplus_i e_iR$ is the $n_k \times n_k$ matrix ring over the local ring e_kRe_k where n_k is the number of e_iR appearing in $\oplus_i e_iR$. Since a finite direct sum of q-hypercyclic rings is q-hypercyclic, the matrix ring is q-hypercyclic. □

Following the same method as for semiperfect q-hypercyclic rings [63], Jain and Saleh obtained the following:

Lemma 8.18. (*Jain and Saleh,* [67]). *Let R be a q-hypercyclic ring with finite right uniform dimension. Then R is right self-injective.*

This lemma is used in the proof of the following theorem.

Theorem 8.19. (*Jain and Saleh,* [67]). *Let R be a q-hypercyclic ring with Krull dimension. Then R is artinian.*

Proof. There exists a prime ideal P of R such that $K(R/P) = K(R)$. Since R is q-hypercyclic, $S = R/P$ is a q-hypercyclic ring ([63], Lemma 2.6). Therefore, S is right self-injective; that is, $E(S) = S$. Since S is a prime Goldie ring, $Q(S) = E(S) = S$. Thus S is artinian, that is, $K(S) = 0$, which gives R is artinian. □

If R is a ring with Krull dimension such that the injective hull of every cyclic R-module is finitely generated or the quasi-injective hull of every cyclic R-module is finitely generated, it is not known whether R is artinian or not.

Remark 8.20. Let R be a ring with Krull dimension, and let P be a minimal prime ideal of R such that the prime ring R/P has a left classical quotient ring (In particular, if R has also Krull dimension as a left R-module). Then, if each cyclic R-module has a finitely generated quasi-injective hull, R must be artinian.

We recall that a module M over a ring R is called π-injective (also called quasi-continuous) if for every pair of R-submodulcs N_1, N_2 of M with $N_1 \cap N_2 = 0$ each projection π: $N_1 \oplus N_2 \to N_i$, $i = 1, 2$, can be lifted to an endomorphism of M.

Goel and Jain [34] considered rings with finite uniform dimension such that any cyclic R-module is π-injective, or more generally has cyclic π-injective hull. These results generalize the results for semiperfect rings over which each cyclic R-module has injective or quasi-injective hull.

Recall that if $K = Hom_R(E(A), E(A))$, then the $q.inj.hull(A) = KA$ and the π-injective hull of A, denoted by $\pi(A) = VA$, where V is the subring of K generated by the idempotents of K.

Lemma 8.21. (*Goel and Jain,* [34]). *Let M be π-injective and $E(M) = \oplus A_i$, be a direct sum of submodules. Then $M = \oplus(A_i \cap M)$.*

Lemma 8.22. *Let R be a ring with finite uniform dimension. If the π-injective hull of R is cyclic, then R is π-injective.*

Proof. See [34]. □

Remark 8.23. A ring homomorphic image of a π-hypercyclic ring is also a π-hypercyclic ring.

Lemma 8.24. *Let R be a ring with finite uniform dimension. Then R is π-hypercyclic if and only if $R = e_1R \oplus ... \oplus e_nR$, where e_iR are valuation rings.*

Proof. See [34]. Only if part: The proof follows from the fact that each e_iR is uniform and for every submodules A and B of e_iR, $e_iR/A \cap B$ is also uniform. □

The proof of the following is straightforward.

Lemma 8.25. *(a) Let A be essential in an injective module B. Then the π-injective hull of $A \times B$, $\pi(A \times B) = E(A) \times B$.*

(b) Let R be an artinian ring with radical J and let I he a right ideal of R. If $R/I = \oplus_{i=1}^{n} N_i$ then the composition length of $R/J \geq n$.

A ring R is called right π-hypercyclic if each cyclic right R-module has cyclic π-injective hull. Following the techniques given earlier for rings over which each proper cyclic module is continuous it follws that each proper cyclic R-module is π-injective if and only if each proper cyclic R-module is continuous where R is a semiperfect nonlocal ring with nil radical.

Jain and Saleh [68] called a ring π-hypercyclic if each cyclic module has a cyclic π-injective hull. They studied π-hypercyclic rings with finite uniform dimension and proved the following.

Theorem 8.26. (*Jain and Saleh,* [68]). *Let R be a π-hypercyclic indecomposable ring with finite uniform dimension other than* 1*. Then $R = M_n(A)$, where A is a right valuation ring, $n > 1$ if and only if R is self-injective.*

Proof. Let R be self-injective. Thus there exist primitive orthogonal idempotents e_i, $1 \leq i \leq n$ are primitive idempotents such that $R = e_1R \oplus ... \oplus e_nR$. Let $\alpha : e_1R \to e_2R$ be a non-zero R-homomorphism. Then $R/I \cong e_2R \times e_2R \times ... \times e_nR$ for some right ideal I of R. Thus R/I is projective and hence $R = I \oplus K$ where

$K \cong e_2R \times e_2R \times ... \times e_nR$. Since R is self-injective, Azumaya Diagram gives $e_1R \cong I \times e_2R$ which forces I to be zero as e_1R is indecomposable. Thus $e_1R \cong e_2R$. Let $[e_kR] = \Sigma e_iR$, $e_iR \cong e_kR$. Since R is indecomposable, $R = [e_1R]$. Therefore, R is a matrix ring over a local ring $A \cong eRe$ where $e = e_1$. It remains to show that eRe is a right valuation ring. We first show that for any right ideal $I \subset eR$, eR/I is uniform. Then it follows that the submodules of eR are linearly ordered. Let A and B be right ideals of eRe. Then $AeR \subset (eR)^2$ and $BeR \subset (eR)^2$. Since the submodules of eR are linearly ordered, $AeR \subset BeR$ or $BeR \subset AeR$ and so $AeRe \subset BeRe$ or $BeRe \subset AeRe$, that is, $A \subset B$ or $B \subset A$. The converse follows from the fact that an $n \times n$ matrix ring $M_n(A)$, $n > 1$ is self π-injective if and only if it is self-injective. □

9. Cyclic Modules being Direct Sum of Projective, Injective, CS, and Noetherian

The study of rings via decomposition properties of cyclic modules has been considered by many authors. Chatters studied rings whose each cyclic module is a direct sum of a projective module and a noetherian module and proved the following.

Theorem 9.1. (*Chatters,* [16]). *A ring R is right noetherian if and only if every cyclic module is a direct sum of a projective module and a noetherian module.*

In [46] Huynh and Dung showed the following.

Theorem 9.2. (*Huynh and Dung,* [46]). *A ring R is right artinian if and only if each cyclic right R-module is the direct sum of an injective module and a finitely cogenerated module.*

Huynh extended it and proved the following.

Theorem 9.3. (*Huynh,* [43]). *A ring R is hereditarily artinian if and only if each cyclic right R-module is the direct sum of an injective module and a finite module.*

Huynh, Dung and Smith obtained the following.

Theorem 9.4. (*Huynh, Dung and Smith,* [47]). *The following statements are equivalent for a ring R:*

(i) Every right ideal is the direct sum of an injective module and a finitely generated semisimple right ideal.

(ii) Every essential right ideal is the direct sum of an injective module and a finitely generated semisimple right ideal.

(iii) R is a direct sum of minimal right ideals and injective right ideals of length 2.

(iv) Every cyclic right R-module is the direct sum of an injective module and a semisimple module.

(v) Every right R-module is the direct sum of an injective module and a semisimple module.

(vi) R is an artinian serial ring such that $(J(R))^2 = 0$.
(vii) Any of the left-sided analogues of (i)-(v).

Theorem 9.5. (*Osofsky and Smith,* [97]). *A ring R is right noetherian if every cyclic right module is a direct sum of a projective module and an injective module.*

As a consequence, it follows that if each cyclic right R-module is injective or projective then the ring R is right noetherian.

Goel, Jain and Singh [35] had considered rings whose each cyclic module is either injective or projective. They obtained the following.

Theorem 9.6. (*Goel, Jain and Singh,* [35]). *If each cyclic right R-module is injective or projective then $R = A \oplus B$ where A is semisimple artinian and B is a simple right semi-hereditary right Ore-domain whose each proper cyclic module is semisimple.*

Smith in his paper [104] independently proved the same result. The structure of rings whose each cyclic module is a direct sum of a projective module and an injective module was completely described by Huynh in [45].

In 1991, Smith asked the question whether a ring is right noetherian if each cyclic module is a direct sum of a projective module and a module that is either injective or noetherian. This question has been recently answered in the affirmative by Huynh and Rizvi [51].

Theorem 9.7. (*Huynh and Rizvi,* [51]). *A ring R is right noetherian if and only if every cyclic module is a direct sum of a projective module and a module Q where Q is either injective or noetherian.*

This clearly extends the above two results of Chatters [16] and Osofsky-Smith [97].

Holston, Jain and Leroy [42] call a ring R a right WV-ring if each simple right R-module is injective relative to proper cyclics. If R is a right WV-ring, then they show that R is either right uniform or a right V-ring.

Lemma 9.8. *Let R be a WV-ring, and R/A and R/B be proper cyclic modules such that $A \cap B = 0$. Then R is a V-ring.*

Proof. Straightforward. □

Theorem 9.9. *Let R be a WV-ring which is not a V-ring. Then R must be uniform.*

Proof. See ([42], Theorem 2). □

A right R-module M is said to satisfy the property (*) if we can write $M = A \oplus B$, where A is either a CS-module or a noetherian module, and B is a projective module. It was shown by Plubtieng and Tansee [98] that a ring R is right noetherian if and only if every 2-generated right R-module satisfies (*). However, if every cyclic R-module satisfies (*), then R need not be right noetherian.

Under a stronger assumption on a cyclic right module C than the condition (*), namely, if every homomorphic image of C is projective, CS, or noetherian, Holston, Jain and Leroy show that C is noetherian when R is a WV-ring.

Theorem 9.10. (*Holston, Jain and Leroy,* [42]). *Let C be a cyclic R-module such that each homomorphic image of C is either CS, noetherian, or projective.*

(a) Then C has finite uniform dimension.

(b) If R is a WV-ring, then C is noetherian.

Proof. See ([42], Theorem 11). □

Lemma 9.11. (*Holston, Jain and Leroy,* [42]).

(a) Let C be an R-module and $S = Socle(C)$. If C/S is a uniform R-module, then for any two submodules A and B of C with $A \cap B = 0$, either A or B is semisimple. Furthermore, if $C/I = A/I \oplus B/I$ is a direct sum with B/I a projective module, then $C = A \oplus B'$, where $B = B' \oplus I$.

(b) Let R be a WV-ring. Let C be a cyclic module with a projective socle (equivalently, $S = Socle(C)$ is embeddable in R). If C/S is a uniform R-module and each homomorphic image of C satisfies (*), then C is noetherian.

Proof. See ([42], Lemmas 12-14). □

Theorem 9.12. (*Holston, Jain and Leroy,* [42]).

(a) Let R be a V-ring. Let M be a finitely generated R-module with projective socle. Suppose each subfactor of M satisfies (*). Then M is noetherian, and $M = X \oplus T$ where X is semisimple and T is noetherian with zero socle. In particular, if R is a V-ring such that each cyclic module satisfies (*), then $R = S \oplus T$, where S is semisimple artinian and T is a finite direct sum of simple noetherian rings with zero socle.

(b) For a WV-ring R, R is noetherian if and only if each cyclic R-module satisfies (*).

The following question of Camillo and Krause remains open:

Is a ring R right noetherian if for any nonzero right ideal A of R, $R/A \ncong R$ is an artinian right R-module?

10. Cyclic Modules Embeddable (Essentially) in Free Modules

It is well-known that a ring R is quasi-Frobenius (QF for short) if and only if each right R-module embeds in a projective or, equivalently, in a free module. More generally, a ring R is called a right FGF ring if each finitely generated right R-module embeds in a free module.

It is known that if R is both left and right FGF then R is, again, a QF ring, but it is an open problem whether a right FGF ring is QF. This problem appeared first in Levy's paper [80] as a question for right Ore rings, and it was later formulated in the present form by Faith. Osofsky proved that a right PF ring is semiperfect and has finite essential right socle [92]. Using this, Bjork [9] proved that a right FGF right self-injective ring must be QF. This was also obtained, independently, by Tolskaya (cf. [30]). Menal [83] used a modification of Osofsky's arguments to prove that if each cyclic right R-module embeds in a free module and the injective hull $E(R_R)$ is projective then R is QF. Jain and Lopez-Permouth studied rings under a tighter embedding hypothesis, more specifically, rings whose cyclic modules are essentially embeddable in projective modules (direct summands of R_R) [58]. Such rings are called right CEP (right CES rings). Examples of right CEP rings include QF-rings and right uniserial rings. Indeed a ring R is a QF-ring if and only if R is both a right and left CEP-ring. Jain and Lopez-Permouth [58] provided the following characterization of QF-rings.

Theorem 10.1. (*Jain and Lopez-Permouth,* [58]). *For an arbitrary ring, the following are equivalent:*

(a) R is QF.

(b) R is CEP and QF-3.

(c) Every cyclic R-module has a projective injective hull.

For a semiperfect CEP ring, Jain and Lopez-Permouth proved the following.

Theorem 10.2. (*Jain and Lopez-Permouth,* [58]). *A semiperfect ring R is CEP if and only if the following hold:*

(a) R is right artinian.

(b) Every indecomposable projective module is uniform, and

(c) Every indecomposable projective module is weakly R-injective.

The following example due to Jain and Lopez-Permouth [58] is an example of a local CEP ring which is neither right uniserial nor quasi-Frobenius.

Example. (Jain and Lopez-Permouth, [58]). Let S be a ring having only three right ideals, namely, (0), $J(S)$ and S and not right self-injective. Let $R = S \propto S$ be the trivial extension of S by itself. Then R is a local CEP ring which is neither right uniserial nor quasi-Frobenius.

In [3] Al-Huzali, Jain and Lopez-Permouth asked if each right CEP ring (and hence right CES ring) is semiperfect. This question was answered in the affirmative by Gomez-Pardo and Guil Asensio [37]. Gomez-Pardo and Guil Asensio first proved the following.

Theorem 10.3. (*Gomez-Pardo and Guil Asensio,* [37]). *Let R be a ring and P_R a finitely generated projective module. Suppose that $\Omega(R)$ denotes a set of representatives of the isomorphism classes of simple right R-modules and $C(P)$ denotes a set of representatives of the isomorphism classes of simple submodules of P. Assume that $|\Omega(R)| \leq |C(P)|$ and that every cyclic submodule of $E(P_R)$ is essentially embeddable in a projective module. Then P_R cogenerates the simple right R-modules and has finite essential socle.*

In particular, the above theorem says that if R is a ring such that $E = E(R_R)$ is a cogenerator and every cyclic submodule of E_R is essentially embeddable in a projective module, then R_R has finite essential socle. The proof given by Gomez-Pardo and Guil Asensio involves an adaptation of Osofsky's counting argument.

As a consequence of the above theorem, Gomez-Pardo and Guil Asensio obtained the following which answers in the affirmative the question asked by Al-Huzali, Jain and Lopez-Permouth.

Theorem 10.4. (*Gomez-Pardo and Guil Asensio,* [37]). *Every right CEP ring is right artinian. In particular, every right CES ring is right artinian.*

Proof. It follows from the above theorem that R_R has finite essential socle. Since every cyclic right R-module embeds in a finitely generated free module, it has also finite essential socle. It is well-known that this implies R is right artinian. □

As a consequence of this result, all results obtained in ([58], [57], [3]) for semiperfect right CEP or CES ring hold for all right CEP and CES ring. In [57] Jain and Lopez-Permouth had shown the structure of semiperfect right CES ring. In view of the result of Gomez-Pardo the result of Jain and Lopez-Permouth gives the structure of any right CES ring.

Theorem 10.5. ([57], [37]). *Let R be a ring. Then the following conditions are equivalent:*

(i) R is right CES.
(ii) R is of one of the following types:
(a) R is (artinian) uniserial as a right R-module,
(b) R is an $n \times n$ matrix ring over a right self-injective ring of type (a), or
(c) R is a direct sum of rings of types (a) or (b).

11. Restricted Regular Rings and Restricted Artinian Rings

A ring R is said to satisfy the right restricted minimum condition if for every proper right ideal I of R, R/I is artinian as a right R-module. Cohen studied commutative rings with restricted minimum condition [20] and proved the following.

Theorem 11.1. (*Cohen,* [20]). *A commutative ring R satisfies the restricted minimum condition if and only if R is noetherian and every proper prime ideal is maximal.*

Chatters studied hereditary noetherian rings with restricted minimum condition and proved the following.

Theorem 11.2. (*Chatters,* [17]).

(i) Let R be a left Noetherian left hereditary ring, and let I be a finitely generated essential right ideal of R. Then R satisfies the descending chain condition for finitely generated right ideals which contain I.

(ii) Let R be a hereditary Noetherian ring. Then R satisfies both the left as well as right restricted minimum condition.

The following example, due to Small [102], shows that a left hereditary left noetherian ring does not necessarily satisfy the left restricted minimum condition.

Example. (Small, [102]). Let R be the ring of all matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where a is an integer and b and c are rationals. This ring R is left noetherian and left hereditary. Let I be the ideal of R consisting of all matrices of the form $\begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix}$.
I is an essential left ideal of R, but R/I is a ring isomorphic to the ring of integers, and it follows that R/I does not satisfy the minimum condition for left R-submodules.

Jain and Saroj Jain studied rings whose each proper homomorphic image is a von Neumann regular ring and called such rings restricted regular rings. They proved the following.

Theorem 11.3. (*Jain and Saroj Jain,* [56]). *Let R be a nonprime right noetherian ring. Then R is a restricted regular ring if and only if*

(i) R is semisimple artinian, or

(ii) R has exactly one non-trivial ideal, namely, the Jacobson radical $J(R)$, and is isomorphic to an $n \times n$ matrix ring over a local ring, or

(iii) R has exactly three non-trivial ideals, namely, $J(R)$, $ann_l(J(R))$ and $ann_r(J(R))$ and is isomorphic to $\begin{pmatrix} U & N \\ 0 & V \end{pmatrix}$, where U, V are simple artinian and N is an irreducible U-V bimodule.

Theorem 11.4. (*Jain and Saroj Jain,* [56]). *If R is a prime right noetherian restricted regular ring then R is semisimple and each non-trivial ideal is a unique product of maximal ideals.*

The next result characterizes right duo restricted regular rings without chain conditions.

Theorem 11.5. (*Jain and Saroj Jain,* [56]). *A right duo ring R is a restricted regular ring if and only if R is strongly regular or R has exactly one proper ideal.*

Almost perfect domains, that is, integral domains whose proper homomorphic images are perfect were introduced by Bazzoni and Salce [5] in connection with the study of the existence of strongly flat covers over commutative integral domains [8]. Since a one-dimensional Noetherian domain is an almost perfect domain, it is natural to look for conditions ensuring that an almost perfect domain is Noetherian. Salce raised a question as to when any ring, not necessarily commutative domain, is an almost perfect ring?

In a recent work Abuhlail-Jain-Laradji [1] showed, among other results, that such rings are either perfect or prime and radical of prime local ring is nil.

12. Questions, Exercises and Open Problems

1. (Koethe, [77]) Describe rings over which each right and left module is a direct sum of cyclic modules.
2. Is every right PCI-domain also a left PCI-domain?

   The only result known in this direction is the one due to Boyle and Goodearl which says that a left and right noetherian domain is a left PCI-domain if and only if it is a right PCI-domain.
3. Let $R = K[t, \sigma, \delta]$ be a twisted differential polynomial ring over a division ring K. Suppose R is a left V-domain. Then R is a right PCI-domain if and only if σ is onto. Does there exist an example when R is a left V-domain with σ not onto (see Theorem 6.2, [60]).
4. (Camillo and Krause) Is a right Ore domain D necessarily right noetherian if every cyclic right D-module is projective or artinian? This is equivalent to: *Is a ring R right noetherian if for any nonzero right ideal A of R, R/A is an artinian right R-module*?
5. (Faith, [29]) Characterize a right Ore domain whose every proper cyclic module C is injective modulo its annihilator ideal.
6. (Faith, [31]) Is every right CSI ring right noetherian?

   Faith has shown that a right CSI ring is right noetherian under some additional conditions [31].
7. Describe a ring over which each cyclic module is a proper homomorphic image of an injective module.
8. Is a prime ring whose each cyclic module is quasi-continuous a right nonsingular ring?
9. Is it true that a cyclic module whose quotients are CS, a finite direct sum of uniform modules? (this is true for projective modules)
10. (Faith, [30]) Is every right FGF ring also a QF ring?

    A ring R is called a right FGF ring if each finitely generated right R-module embeds in a free module. It is known that if R is both left and right FGF then R is a QF ring. Bjork [9] proved that a right FGF right self-injective ring must be QF. This was also obtained, independently, by Tolskaya (cf. [30]).
11. Describe non-simple prime right noetherian ring R such that each proper ring homomorphic image is von Neumann regular.
12. Is it true that a local hypercyclic ring R has a nil radical?
13. Study a prime right noetherian restricted regular ring. Is it necessarily artinian?
14. Is every right $PCQI$-domain right noetherian? It is known to be true for simple rings and for V-rings.
15. (Salce, [100]) Study rings such that each R-homomorphic image of R_R is a perfect module.

Commutative rings with this hypothesis has been studied by Salce in [100]. It has been shown by Abuhlail-Jain-Laradji [1] that such rings are either prime or perfect. In case it is prime local or non-domain then the radical is nil.

16. Study rings over which every proper quotient of M is a finitley presented (i) completely *pqc*-module, or (ii) completely pure injective module. See page 21 for the definition of completely *pqc*-module. Gomez-Pardo and Guil-Asensio showed that if M is a finitely presented completely *pqc*-module, then M has ACC (and DCC) on direct summands and so it is a (finite) direct sum of indecomposable modules (see [38],Theorem 2.9).
17. (Dinh, Guil Asensio, Lopez-Permouth, [27]) Let E be a finitely generated module such that any pure quotient is pure-injective. Is E a direct sum of indecomposable pure-injective modules?

    Dinh, Guil Asensio and Lopez-Permouth [27] showed that if R is a ring of cardinality at most $2^{\aleph_0}$ and E is a countably generated injective right R-module such that every quotient of E is injective then E is a direct sum of indecomposable modules.

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S. K. Jain
Department of Mathematics, Ohio University, Athens, Ohio-45701, USA
e-mail: `jain@math.ohiou.edu`

Ashish K. Srivastava
Department of Mathematics and Computer Science, St. Louis University, St. Louis, MO-63103, USA
e-mail: `asrivas3@slu.edu`