Rings are associative, not necessarily with 1.

If R is a ring then R^1 denotes the standard extenion of R to a unital ring.

For a ring R let $\mathsf{P}(R)$ be the prime radical of R, and $\mathsf{N}(R)$ be the upper nil radical of R.

A ring is reduced if it has no nontrivial nilpotent elements or, equivalently, if it satisfies the following quasi-identity:

$$x^2 = 0 \Rightarrow x = 0. \tag{1}$$

Theorem 1 (Thierrin and others) *For a ring R the following conditions are equivalent:*

1. *R is reduced;*
2. *R is semiprime and all minimal prime ideals of R are completely prime;*
3. *R is a subdirect product of domains;*
4. *All subrings of R are semiprime.*

Theorem 2 *Let R be a reduced ring. Then for any $n > 1$ and every $\sigma \in S_n$ the following quasi-identity is satisfied:*

$$x_1 \cdots x_n = 0 \Rightarrow x_{\sigma(1)} \cdots x_{\sigma(n)} = 0. \qquad (2)$$

In particular, for every $x, y, z \in R$ we have:

$$xy = 0 \Rightarrow yx = 0, \quad xyz = 0 \Rightarrow xzy = 0$$

$$\text{and} \quad xy = 0 \Rightarrow xzy = 0.$$

A ring R is:

<u>Reversible</u> if $ab = 0 \Rightarrow ba = 0 \, \forall a, b \in R$;

<u>Symmetric</u> if $abc = 0 \Rightarrow acb = 0 \, \forall a, b, c \in R$;

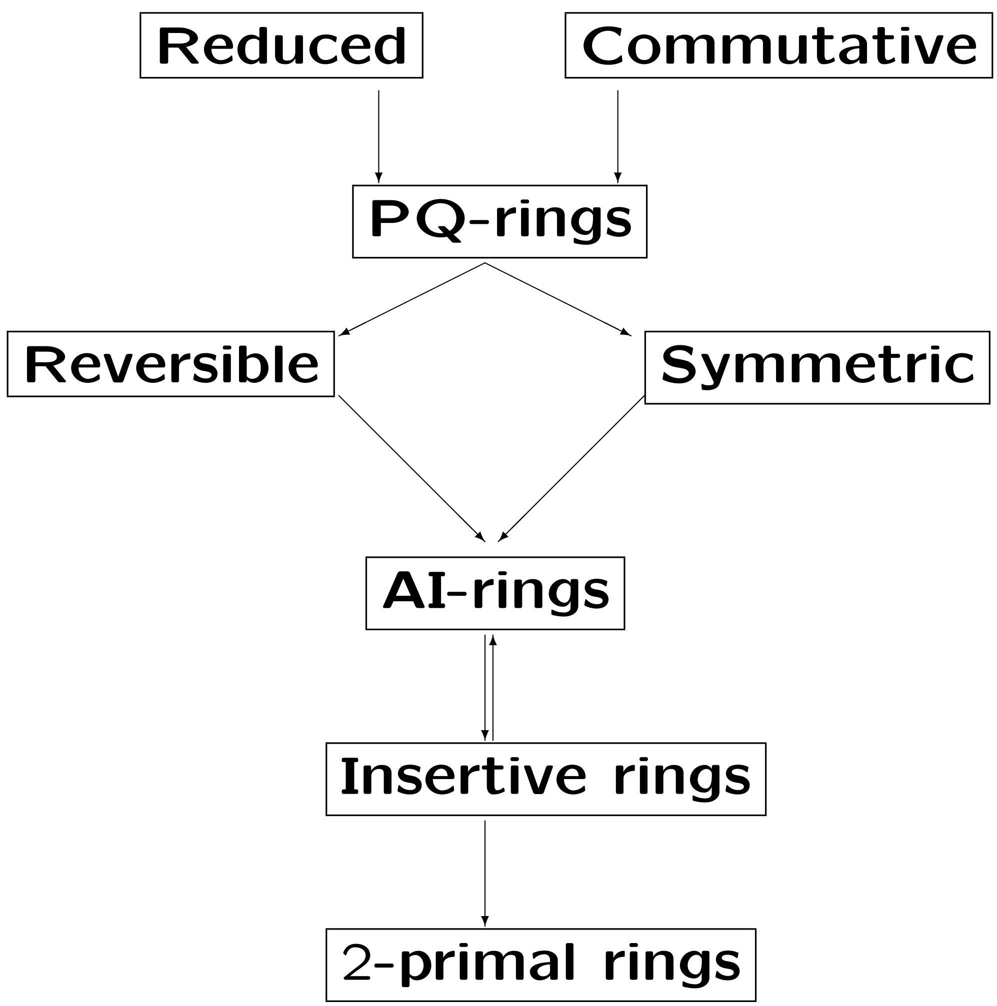
<u>PQ-ring</u> if $a_1 \cdots a_n = 0 \Rightarrow a_{\sigma(1)} \cdots a_{\sigma(n)} = 0$ $\forall n > 1 \, \forall a_1, \ldots, a_n \in R$ and $\forall \sigma \in S_n$;

<u>Insertive</u> if $ab = 0 \Rightarrow acb = 0 \, \forall a, b, c \in R$;

<u>AI-ring</u> if annihilators in R are ideals;

<u>2-primal</u> if $R/\mathrm{P}(R)$ is reduced.

Scheme of inclusions for classes:



Theorem 3 *Let* R *be an AI-ring and* $\mathsf{P}(R)$ *=P. If* $r \in R$ *and* $r^n = 0$ *then* $(R^1 r R^1)^n = 0$*. In particular,* $\mathsf{N}(R) = P$ *and is a sum of nilpotent ideals. Moreover,* R/P *is a reduced ring.*

Remark A ring R is 2-primal iff for every $r, s \in R$ we have $rs = 0 \Rightarrow sr \in \mathsf{P}(R)$.

In the sequel, F represents a commutative, unital ring (basic ring).

Modules = F-modules;

Algebras = algebras over F (F-algebras);

$\mathcal{L}$ = the class of all modules of finite length;

$\mathcal{D}$ = the class of all modules with DCC condition on submodules;

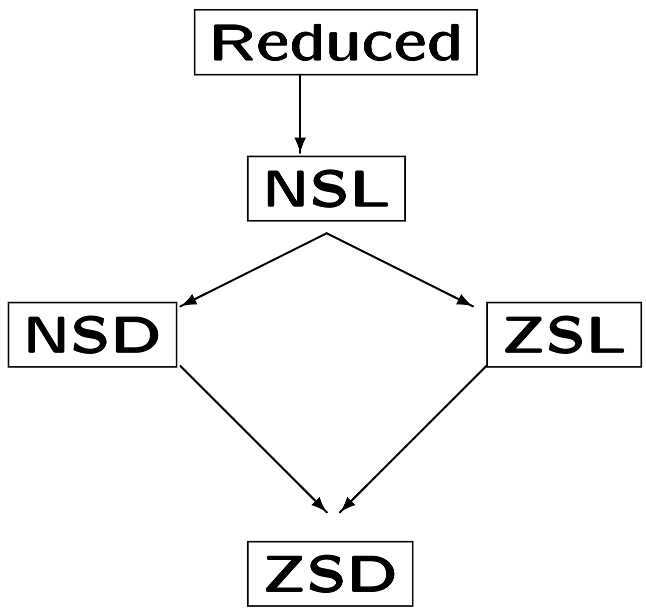
Zero subalgebra = subalgebra with zero (trivial) multiplication.

Clearly, rings are $\mathbb{Z}$-algebras.

An algebra A is a ZSL-algebra if every zero subalgebra of A belongs to $\mathcal{L}$ and A is an NSL-algebra if every nilpotent subalgebra of A belongs to $\mathcal{L}$.

Similarly, A is a ZSD-algebra if every zero subalgebra of A belongs to $\mathcal{D}$ and A is an NSD-algebra if every nilpotent subalgebra of A belongs to $\mathcal{D}$.

Scheme of inclusions for classes:



Lemma 1 *Let $\mathcal{C}$ be one of the discussed classes and let A be an algebra.*

1. *If $A \in \mathcal{C}$ and $B \subseteq A$ is a subalgebra, then $B \in \mathcal{C}$.*
2. *If $I \subseteq A$ is an ideal such that I and A/I belong to $\mathcal{C}$ then A belongs to $\mathcal{C}$.*
3. *The class $\mathcal{C}$ is closed under forming finite direct sums, but is not closed under going to homomorphic images.*

Example 1 (Laffey) *Let F be a formally real field, V be a linear space with basis $\{v_0, v_1, v_2, \ldots\}$ and with multiplication given by*

$$v_0^2 = 0 \quad \textit{and} \quad v_i v_j = \delta_{ij} v_0 \tag{3}$$

for $(i, j) \neq (0, 0)$. Then Fv_0 is the only non-trivial zero subalgebra of V, $V^3 = 0$ and V is a ZSL-algebra and ZSD-algebra, but not an NSL-algebra and even not an NSD-algebra.

Theorem 4 *Let A be a semiprime algebra. Then the following conditions are equivalent:*

1. *A is a ZSD-algebra;*
2. *A is a ZSL-algebra;*
3. *A is an NSD-algebra;*
4. *A is an NSL-algebra;*
5. *$A = B \oplus C$, where $B \in \mathcal{L}$ and C is a reduced algebra.*

Theorem 5 *Let A be a ZSD-algebra. Then every nil subalgebra of A is nilpotent. In particular:*

1. $\mathsf{N}(A) = \mathsf{P}(A)$;
2. *If A is an NSD-algebra, then* $\mathsf{P}(A) \in \mathcal{D}$ *and* $A/(\mathsf{P}(A))$ *is an NSD-algebra;*
3. *If A is an NSL-algebra, then* $\mathsf{P}(A) \in \mathcal{L}$ *and* $A/(\mathsf{P}(A))$ *is an NSL-algebra.*

Lemma 2 *Let* $A \in \mathcal{D}$ *be an algebra and* I *be the annihilator of* A *in* F.

If A *is prime, then* F/I *is a field and* A *is a simple, unital algebra, finite dimensional over* F/I.

If A *is semiprime then* A *is a finite direct sum of simple algebras. Thus* $A \in \mathcal{L}$.

If A *is a nil algebra then* A *is nilpotent.*

If $S \subseteq A$ is a subset, then

$$\mathrm{Ann}_A(S) = \{x \in A \mid xS = Sx = 0\}. \qquad (4)$$

If modules $M, N \in \mathcal{L}$ then $\mathrm{Hom}_F(M, N) \in \mathcal{L}$.

Lemma 3 *Let $B \subseteq A$ be an ideal of an algebra A and $C = \mathrm{Ann}_A(B)$. If $B \in \mathcal{L}$ then $A/C \in \mathcal{L}$.*

Lemma 4 *Let $B \subseteq A$ be algebras and I be the largest ideal of A contained in B. If the module $A/B \in \mathcal{L}$, then $A/I \in \mathcal{L}$.*

Let A be an algebra. By $\Delta(A)$ we denote the sum of all the ideals I of A such that $I \in \mathcal{D}$.

Lemma 5 *Let A be an algebra and $a \in A$. The following conditions are equivalent:*

1. *$a \in \Delta(A)$;*
2. *The ideal $A^1 a A^1$ belongs to $\mathcal{D}$;*
3. *The ideal $A^1 a A^1$ belongs to $\mathcal{L}$.*

Lemma 6 *Let A be an algebra, $B \subseteq A$ a subalgebra and $I \subseteq A$ an ideal of A. Then*

1. $\Delta(\Delta(A)) = \Delta(A)$;
2. $B \cap \Delta(A) \subseteq \Delta(B)$;
3. *If the module* $A/B \in \mathcal{L}$, *then* $B \cap (fccA = \Delta(B)$;
4. $(\Delta(A) + I)/I \subseteq \Delta(A/I)$;
5. *If* $\Delta(A) \in \mathcal{D}$, *then* $\Delta(A/(\Delta(A))) = 0$.

Lemma 7 *Let A be a semiprime algebra with the socle* $\text{soc}(A) = \oplus_{s\in S} I_s$, *where I_s are homogeneous components. Then* $\Delta(A)$ *is the sum of all ideals $I_s \subseteq \mathcal{D}$. In particular, if* $\Delta(A) \in \mathcal{D}$, *then* $\Delta(A)$ *has a unit and* $A = \Delta(A) \oplus B$, *where B is an ideal with* $\Delta(B) = 0$.

Theorem 6 *Let* A *be a ZSD-algebra. Then the algebra* $A/(\Delta(A))$ *is reduced. In particular,* $\Delta(A)$ *contains all nilpotent elements of* A *and* $N = \Delta(N)$ *for every nil subalgebra* $N \subseteq A$.

Lemma 8 *Let* N *be a nil algebra,* I *be a maximal zero ideal of* N *and* $J = Ann_N(I)$. *Then:*

1. $J^3 = 0$;
2. *If* N *is a ZSD-algebra, then* N *is nilpotent;*
3. *If* $I \in \mathcal{L}$ *then* $N/J \in \mathcal{L}$.

A field F is <u>square dependent</u> if for any infinite sequence $a_1, a_2, \ldots$ of elements from F there exists $n \geq 1$ such that the equation

$$\sum_{i=1}^{n} a_i x_i^2 = 0 \qquad (5)$$

has a nontrivial solution in F.

Fields being not square dependent are <u>square independent</u>.

Every algebraically closed field is square dependent, while every formally real field is square independent.

Example 2 (Laffey) Let F be a square independent field with elements $a_0 = 0, a_1, a_2, \ldots$ such that the equation $\sum_{i=1}^{n} a_i x_i^2 = 0$ has only the zero solution in F, for $n = 1, 2, \ldots$. Let V be a vector space over F, with basis $\{v_0, v_1, v_2, \ldots\}$. Then V is an algebra under multiplication given by:

$$v_i v_j = \delta_{ij} a_i v_0 \quad \text{for} \quad i, j \geq 0. \tag{6}$$

We have $V^3 = 0$, $\Delta(V) = V$ and V is a ZSL-algebra, but not an NSL-algebra.

Theorem 7 (Laffey) *Let F be a field. Then F is square dependent if and only if every ZSL-algebra is an NSL-algebra.*

Example 3 Let K be a field and let $F = K(t_1, t_2, \ldots)$ be the field of rational functions of infinitely many indeterminates $t_1, t_2, \ldots$. Then, using the infinite sequence $(t_1, t_2, \ldots)$ one can see that F is square independent. However, for any algebraically closed field K, F is not formally real.

An arbitrary ring F is square dependent if F/M is a square dependent field for every maximal ideal M of F.

Rings being not square dependent will be called square independent.

Example 4 Let K be a field and let $F = K[t_1, t_2, \ldots]$ be the ring of polynomials of infinitely many commuting indeterminates $t_1, t_2, \ldots$.
If K is countable then F is square independent, by Example 2.
If K is uncountable and algebraically closed, then F is square dependent, while $\mathrm{Q}(F) = K(t_1, t_2, \ldots)$ is square independent.

Theorem 8 *For our basic ring F the following conditions are equivalent:*

1. *F is square dependent;*
2. *$ZSL = NSL$;*
3. *Any nil algebra with property ZSL belongs to $\mathcal{L}$.*

Theorem 9 *Let F ba a square dependent ring and A be a ZSL-algebra. Then $\mathsf{P}(A) \in \mathcal{L}$ and it is a nilpotent ideal. Moreover, A/I is a ZSL-algebra for every ideal $I \subseteq \mathsf{P}(A)$.*

Question 1 *Can versions of the two above theorems be proved for ZSD-algebras, NSD-algebras and the class $\mathcal{D}$?*

Let, as in Lemma 8, N be a nil algebra, I a maximal zero ideal of N and $J = \mathrm{Ann}_N(I)$.

Question 2 *Let $I \in \mathcal{D}$. Does N/J belong to $\mathcal{D}$?*

Question 3 *Let F be square dependent, and $I \in \mathcal{D}$. Does J belong to $\mathcal{D}$?*

Jan Krempa, *Some examples of reduced rings,* Algebra Colloqium 3(1996), 289-300.

—, *On finiteness conditions for subalgebras with zero multiplication,* Colloq. Math. 102(2005) 123-135.

—, *On nil-algebras and an invariant of fields,* Tatra Mountain Math. Publ. 32(2005), 63-70.