Double Ore Extensions versus Iterated Ore Extensions

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- $M_{n\times m}(A)$ means the linear space of all $n \times m$ matrices over A.

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Theorem

Let R be a ring with an endomorphism σ and σ-derivation d. Then multiplication in R and the rule $xa = \sigma(a)x + d(a)$, for $a \in R$, define the structure of an Ore extension on the left free R-module with basis $1, x, x^2, \dots$

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3. *σ is injective and $xR + R = Rx + R$;*
4. *$xR + R = Rx + R$ is a free right R-module with basis $1, x$.*

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$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0$$

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where $p_{12}, p_{11} \in K$ and $\tau_1, \tau_2, \tau_0 \in A$;
3. B is a free left A-module with basis $\{y_1^{n_1} y_2^{n_2} : n_1 \geq 0, n_2 \geq 0\}$;
4. $y_1 A + y_2 A + A \subseteq A y_1 + A y_2 + A$.

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- $y_1 A + y_2 A + A = A y_1 + A y_2 + A$.

Commutation formulae

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} a = \sigma(a) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(a), \text{ for all } a \in A$$

where

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : A \to M_{2\times 2}(A)$$

and

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} : A \to M_{2\times 1}(A)$$

In the case B is a right double extension of A, we will write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p_{12}, p_{11}\} \subseteq K$ and $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq A$ and σ, δ are as described above. The set P is called a parameter and τ a tail.

Theorem

Given an K-algebra A suppose that σ is a homomorphism from A to $M_{2\times 2}(A)$, δ is a σ-derivation from A to $M_{2\times 1}(A)$, P is a given set $\{p_{12}, p_{11}\}$ of elements of K and τ is the set $\{\tau_0, \tau_1, \tau_2\}$ of elements of A. Then the associative algebra B generated by A and by y_1, y_2 subject to the relations from the definition of right double extension is a right double if and only if the maps σ_{ij}, $\sigma_{i0} = \delta_i$, $i, j \in \{1, 2\}$ satisfy six relations (R3.1)-(R3.6) from [J.J.Zhang, J.Zhang], page 2674.

Proposition

If $B = K_P[y_1, y_2; \sigma, \delta, \tau]$, then $B = K[x_1][x_2; \sigma_2, d_2]$ is an iterated Ore extension, where σ_2 is a K-linear endomorphism of the polynomial ring $K[x_1]$ defined by $\sigma_2(x_1) = p_{12}x_1 + \tau_2$ and d_2 is a K-linear σ_2-derivation of $K[x_1]$ given by $d_2(x_1) = p_{11}x_1^2 + \tau_1 x_1 + \tau_0$. Moreover B is a double extension of K if and only if $p_{12} \neq 0$.

Proposition

Let $P = \{p_{12}, p_{11}\} \subseteq K$ *and* $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq K$.
$\sigma \colon A \longrightarrow M_{2\times 2}(A)$ *be a homomorphism and* $\delta \colon A \longrightarrow M_{2\times 1}(A)$ *a* σ*-derivation of algebras. Then the following conditions are equivalent:*

1. *the right double extension* $A_P[y_1, y_2; \sigma, \delta, \tau]$ *exists;*

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1. *the right double extension* $A_P[y_1, y_2; \sigma, \delta, \tau]$ *exists;*
2. *one can extend the multiplication from* A *and* C *to the multiplication in the linear space* $A \otimes_K C$*, where* $C = K[x_1][x_2; \sigma_2, \delta_2]$ *is as in Proposition above, by requiring that* $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} a = \sigma(a) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \delta(a)$*, for all* $a \in A$.

Proposition

Suppose $A_P[y_1, y_2; \sigma, \delta, \tau]$ *is given. Then* σ *is invertible iff* $Ay_1 + Ay_2 + A = y_1A + y_2A + A$ *and is free as right and left* A*-module with basis* $1, y_1, y_2$*.*

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Question

Is $A_P[y_1, y_2; \sigma, \delta, \tau]$ a double extension if σ is invertible and $p_{12} \neq 0$?

Theorem

Let A, B be K-algebras such that B is an extension of A and $P = \{p_{12}, p_{11}\} \subseteq K$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq A$, σ is an algebra homomorphism from A to $M_{2\times 2}(A)$ and δ is a σ-derivation from A to $M_{2\times 1}(A)$. The following conditions are equivalent:

- *$B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of A which can be presented as an iterated Ore extension $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$;*

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- *$B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension with $\sigma_{12} = 0$;*

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- *$B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension with $\sigma_{12} = 0$;*
- *$B = A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ such that*

$$\begin{aligned} \sigma_2(A) \subseteq A, &\quad \sigma_2(y_1) = p_{12}y_1 + \tau_2, \\ \delta_2(A) \subseteq Ay_1 + A, &\quad \delta_2(y_1) = p_{11}y_1^2 + \tau_1 y_1 + \tau_0 \end{aligned}$$

where $p_{ij} \in K$ and $\tau_i \in A$.

If one of the equivalent statements holds, then:

- The maps $\sigma = \begin{bmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$, $\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$, and σ_i, δ_i, $i = 1, 2$ are such that:

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$\delta(a) = \begin{bmatrix} d_1(a) \\ d_2(a) - \sigma_{21}(a)y_1 \end{bmatrix}$, for any $a \in A$;
- B is a double Ore extension if and only if $\sigma_1 = \sigma_{11}$ and $\sigma_2|_A = \sigma_{22}$ are an automorphisms of A and $p_{12} \neq 0$.

Theorem

Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ *be a right double extension of the* K*-algebra* A*, where* $P = \{p_{12}, p_{11}\} \subseteq K$*,* $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq A$*,* $\sigma\colon A \to M_{2\times 2}(A)$ *is an algebra homomorphism and* $\delta\colon A \to M_{2\times 1}(A)$ *is a* σ*-derivation. Then,* B *can be presented as an iterated Ore extension* $A[y_2; \sigma_2', d_2'][y_1; \sigma_1', d_1']$ *if and only if* $\sigma_{21} = 0$*,* $p_{12} \neq 0$ *and* $p_{11} = 0$*. In this case,* B *is a double extension if and only if* $\sigma_2' = \sigma_{22}$ *and* $\sigma_1'|_A = \sigma_{11}$ *are automorphisms of* A*.*

Lemma

Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ *be a right double extension,* $k, l \in K$ *and* $0 \neq z = ky_1 + ly_2 \in B$. *Then:*

$$zA \subseteq Az + A \qquad \textit{iff} \qquad kl\sigma_{11} + l^2\sigma_{21} = kl\sigma_{22} + k^2\sigma_{12}.$$

Example

Let $a, b, c \in K$ with $b \neq 0$, $A = K[x]$ and $\sigma\colon A \longrightarrow M_{2\times 2}(A)$ the K-homomorphism given by $\sigma(x) = \begin{bmatrix} 0 & b^{-1}x \\ bx & 0 \end{bmatrix}$. Let $\delta\colon A \to M_{2\times 1}(A)$ denote the σ-derivation determined by the condition $\delta(x) = \begin{bmatrix} cx^2 \\ -bcx^2 \end{bmatrix}$. Then the double extension $B^2(a,b,c) = A_{\{-1,0\}}[y_1, y_2; \sigma, \delta, \{0, 0, ax^2\}]$ exists and it is a K-algebra generated by x, y_1, y_2 subject to the relations:

$$y_2y_1 = -y_1y_2 + ax^2, \quad y_1x = b^{-1}xy_2 + cx^2, \quad y_2x = bxy_1 + (-bc)x^2$$

Proposition

The algebra $B^2 = B^2(a,b,c)$ *has the following properties:*

1. *Suppose that* $\mathrm{char}(K) = 2$*. Then* B^2 *can be presented as iterated Ore extension over* $K[x]$*. In fact* $B^2 = K[x,z][y_2;\sigma_2,d]$*, where* σ_2 *is an automorphism of the polynomial ring* $K[x,z]$ *defined by* $\sigma_2(x) = -x$*,* $\sigma_2(z) = -z$ *and* d *is a* σ_2*-derivation of* $K[x,z]$ *determined by* $d(x) = xz - bcx^2$ *and* $d(z) = bax^2$*.*

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2. B^2 *is a noetherian domain.*

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- Suppose A is a domain. Is also B a domain?
- Suppose R is prime (semiprime noetherian). Is B is also such?

Thanks for your attention