Rings Closed to Semiregular

A. Çiğdem Özcan

Hacettepe University, Ankara, Turkey

joint work with

Pınar Aydoğdu

Lens, 2009

Definitions

$R :=$ a ring with identity
$J :=$ the Jacobson radical of R

$R :=$ a ring with identity
$J :=$ the Jacobson radical of R

Definition

An element $a \in R$ is called regular

$R :=$ a ring with identity
$J :=$ the Jacobson radical of R

Definition

An element $a \in R$ is called regular if there exists $b \in R$ such that $a = aba$.

$R :=$ a ring with identity
$J :=$ the Jacobson radical of R

Definition

An element $a \in R$ is called regular if there exists $b \in R$ such that $a = aba$.
R is called regular if every element of R is regular.
(von Neumann, 1936).

$R :=$ a ring with identity
$J :=$ the Jacobson radical of R

Definition

An element $a \in R$ is called regular if there exists $b \in R$ such that $a = aba$.
R is called regular if every element of R is regular.
(von Neumann, 1936).

These rings were introduced by von Neumann as co-ordinate rings of infinite dimensional projective and continuous geometry.

Rings closed to regular:

Rings closed to regular:

strongly regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

strongly regular

Rings closed to regular:

strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

strongly regular ⇒ strongly π–regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular

Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

Definition 1.1

Idempotents lift modulo J

Definition 1.1

Idempotents lift modulo J if whenever $a^2 - a \in J$ for any $a \in R$,

Definition

Definition 1.1

Idempotents lift modulo J if whenever $a^2 - a \in J$ for any $a \in R$, there exists $e^2 = e \in R$ such that $a - e \in J$.

Semiregular ring:

Theorem 1.2

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

1. R/J is regular and idempotents lift modulo J.

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

1. R/J is regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a regular element d in R such that $a - d \in J$.

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

1. R/J is regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

1. R/J is regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent $e \in aR$ such that $(1 - e)a \in J$.

Semiregular ring:

Theorem 1.2

TFAE for a ring R:

1. R/J is regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent $e \in aR$ such that $(1 - e)a \in J$.
5. For any $a \in R$, there exists an idempotent $e \in Ra$ such that $a(1 - e) \in J$.

(Oberst-Schneider, 1971; Nicholson, 1976; Nicholson-Zhou, 2005)

strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

Definition 2.1

An element of a of R is called (one–sided) unit–regular

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$.

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

R is said to have stable range 1

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

R is said to have stable range 1 if for any a, $b \in R$ satisfying $aR + bR = R$,

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

R is said to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is (right) invertible (Bass, 1964).

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Definition 2.1

An element of a of R is called (one–sided) unit–regular if there exists a (one–sided) unit $u \in R$ such that $a = aua$. R is called (one–sided) unit–regular if every element of R is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

R is said to have stable range 1 if for any a, $b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is (right) invertible (Bass, 1964).

If R is regular, then
R has stable range 1 $\Leftrightarrow$ R is unit–regular (Goodearl, 1979).

Semi Unit–Regular Rings:

Theorem 2.3

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a unit–regular element d in R such that $a - d \in J$.

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a unit–regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a unit–regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a unit–regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a unit–regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent e and a unit b in R such that $e \in aR$, $(1 - e)a \in J$, $ba - (ba)^2 \in J$.

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a unit–regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a unit–regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent e and a unit b in R such that $e \in aR$, $(1 - e)a \in J$, $ba - (ba)^2 \in J$.
5. For any $a \in R$, there exists an idempotent e and a unit b in R such that $e \in Ra$, $a(1 - e) \in J$, $ab - (ab)^2 \in J$.

Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring R:

1. R/J is unit regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a unit–regular element d in R such that $a - d \in J$.
3. For any $a \in R$, there exists a unit–regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent e and a unit b in R such that $e \in aR$, $(1-e)a \in J$, $ba - (ba)^2 \in J$.
5. For any $a \in R$, there exists an idempotent e and a unit b in R such that $e \in Ra$, $a(1-e) \in J$, $ab - (ab)^2 \in J$.
6. R is a semiregular ring with stable range 1.

Definition 2.4

Definition 2.4

A ring R is said to have weak stable range 1

Definition 2.4

A ring R is said to have weak stable range 1 if for any a, $b \in R$ satisfying $aR + bR = R$,

Definition 2.4

A ring R is said to have weak stable range 1 if for any a, $b \in R$ satisfying $aR + bR = R$, there exists an element y in R such that $a + by$ is a one-sided unit (Wu, 1994).

Definition 2.4

A ring R is said to have weak stable range 1 if for any a, $b \in R$ satisfying $aR + bR = R$, there exists an element y in R such that $a + by$ is a one-sided unit (Wu, 1994).

If R is regular, then
R has weak stable range 1 $\Leftrightarrow$ R is one–sided unit–regular.

Semi One–sided Unit–Regular Rings:

Theorem 2.5

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

1. R/J is one-sided unit–regular and idempotents lift modulo J.

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

1. R/J is one-sided unit–regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a one-sided unit regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

1. R/J is one-sided unit–regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a one-sided unit regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
3. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in aRb$ such that $(1-e)a \in J$ and $ba - (ba)^2 \in J$.

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

1. R/J is one-sided unit–regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a one-sided unit regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
3. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in aRb$ such that $(1-e)a \in J$ and $ba - (ba)^2 \in J$.
4. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in bRa$ such that $a(1-e) \in J$ and $ab - (ab)^2 \in J$.

Semi One–sided Unit–Regular Rings:

Theorem 2.5

TFAE for a ring R:

1. R/J is one-sided unit–regular and idempotents lift modulo J.
2. For any $a \in R$, there exists a one-sided unit regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
3. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in aRb$ such that $(1-e)a \in J$ and $ba - (ba)^2 \in J$.
4. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in bRa$ such that $a(1-e) \in J$ and $ab - (ab)^2 \in J$.
5. R is a semiregular ring with weak stable range 1.

strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

R is semi unit–regular ⇔

R is semi unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi unit–regular (H. Chen, M. Chen, 2003).

Theorem 2.6

R is semi one–sided unit–regular ⇔

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that $e_i R e_j$ are semi one–sided unit–regular.

Sketch of the proof

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that $e_i R e_j$ are semi one–sided unit–regular.

Sketch of the proof

(⇐): ⋄ R is semiregular (Chen-Chen, 2003)

Theorem 2.6

R is semi one–sided unit–regular $\Leftrightarrow$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

($\Leftarrow$): $\diamond$ R is semiregular (Chen-Chen, 2003)
$\diamond$ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ R is semiregular (Chen-Chen, 2003)
◇ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
◇ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ R is semiregular (Chen-Chen, 2003)
◇ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
◇ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
◇ All e_iR have the fep (Warfield, 1972).

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ R is semiregular (Chen-Chen, 2003)
◇ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
◇ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
◇ All e_iR have the fep (Warfield, 1972).
◇ All e_iR satisfy outer weak cancellation (Li-Tong, 2002).

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ R is semiregular (Chen-Chen, 2003)
◇ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
◇ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
◇ All e_iR have the fep (Warfield, 1972).
◇ All e_iR satisfy outer weak cancellation (Li-Tong, 2002).

M satisfies outer weak cancellation if $M \oplus K \cong M \oplus L$, then there exist a splitting epimorphisms between K and L (Wu, 1996).

Theorem 2.6

R is semi one–sided unit–regular $\Leftrightarrow$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

($\Leftarrow$): $\diamond$ R is semiregular (Chen-Chen, 2003)
$\diamond$ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
$\diamond$ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
$\diamond$ All e_iR have the fep (Warfield, 1972).
$\diamond$ All e_iR satisfy outer weak cancellation (Li-Tong, 2002).

M satisfies outer weak cancellation if $M \oplus K \cong M \oplus L$, then there exist a splitting epimorphisms between K and L (Wu, 1996).

$\diamond$ $\oplus_{i=1}^{n} e_iR$ satisfies outer weak cancellation. (Cor. 2.11)

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ⋄ R is semiregular (Chen-Chen, 2003)
⋄ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
⋄ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
⋄ All e_iR have the fep (Warfield, 1972).
⋄ All e_iR satisfy outer weak cancellation (Li-Tong, 2002).

M satisfies outer weak cancellation if $M \oplus K \cong M \oplus L$, then there exist a splitting epimorphisms between K and L (Wu, 1996).

⋄ $\oplus_{i=1}^{n} e_iR$ satisfies outer weak cancellation. (Cor. 2.11)
⋄ End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1 (Li-Tong, 2002).

Theorem 2.6

R is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of R such that e_iRe_j are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ R is semiregular (Chen-Chen, 2003)
◇ Aim: End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1.
◇ End$(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
◇ All e_iR have the fep (Warfield, 1972).
◇ All e_iR satisfy outer weak cancellation (Li-Tong, 2002).

M satisfies outer weak cancellation if $M \oplus K \cong M \oplus L$, then there exist a splitting epimorphisms between K and L (Wu, 1996).

◇ $\oplus_{i=1}^{n} e_iR$ satisfies outer weak cancellation. (Cor. 2.11)
◇ End$(\oplus_{i=1}^{n} e_iR)$ has weak stable range 1 (Li-Tong, 2002).
◇ R is semiregular with weak stable range 1.

Definition 3.1

An element a in R is called strongly regular

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.
If any element in R is strongly regular,

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Well-known characterization of strongly regular rings:

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Well-known characterization of strongly regular rings:

R is strongly regular
$\Leftrightarrow$

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2 x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Well-known characterization of strongly regular rings:

R is strongly regular
$\Leftrightarrow$
R is regular and abelian (i.e. every idempotent is central)
$\Leftrightarrow$

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Well-known characterization of strongly regular rings:

R is strongly regular
$\Leftrightarrow$
R is regular and abelian (i.e. every idempotent is central)
$\Leftrightarrow$
R is unit–regular and abelian
$\Leftrightarrow$

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Definition 3.1

An element a in R is called strongly regular if there exists $x \in R$ such that $a = a^2 x$.
If any element in R is strongly regular, then R is called a strongly regular ring (Arens-Kaplansky, 1948).

Well-known characterization of strongly regular rings:

R is strongly regular
$\Leftrightarrow$
R is regular and abelian (i.e. every idempotent is central)
$\Leftrightarrow$
R is unit–regular and abelian
$\Leftrightarrow$
For any $a \in R$, there exist an idempotent e and a unit u in R such that $a = eu$ and $eu = ue$
(Goodearl, 1979)

Semi strongly regular ring:

Theorem 3.2

Semi strongly regular ring:

Theorem 3.2

TFAE for a ring R:

Semi strongly regular ring:

Theorem 3.2

TFAE for a ring R:

1. R/J is strongly regular and idempotents lift modulo J.

Semi strongly regular ring:

Theorem 3.2

TFAE for a ring R:

1. R/J is strongly regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a unit u and an idempotent e in R (resp. in aR) such that $a - eu \in J$ and $eu - ue \in J$.

Semi strongly regular ring:

Theorem 3.2

TFAE for a ring R:

1. R/J is strongly regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a unit u and an idempotent e in R (resp. in aR) such that $a - eu \in J$ and $eu - ue \in J$.
3. For any $a \in R$, there exists a unit u and an idempotent $e \in aR$ such that $(1-e)a \in J$ and $\overline{ua} = \overline{au}$ is an idempotent in R/J.

Semi strongly regular ring:

Theorem 3.2

TFAE for a ring R:

1. R/J is strongly regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a unit u and an idempotent e in R (resp. in aR) such that $a - eu \in J$ and $eu - ue \in J$.
3. For any $a \in R$, there exists a unit u and an idempotent $e \in aR$ such that $(1-e)a \in J$ and $\overline{ua} = \overline{au}$ is an idempotent in R/J.
4. For any $a \in R$, there exists a unit u and an idempotent $e \in Ra$ such that $a(1-e) \in J$ and $\overline{au} = \overline{ua}$ is an idempotent in R/J.

Definition 3.3

A ring R is said to have unit stable range 1

Definition 3.3

A ring R is said to have unit stable range 1 if, for any $a, b \in R$, satisfying $aR + bR = R$, there exists a unit $u \in R$ such that $a + bu$ is a unit. (Goodearl-Menal, 1988).

Definition 3.3

A ring R is said to have unit stable range 1 if, for any $a, b \in R$, satisfying $aR + bR = R$, there exists a unit $u \in R$ such that $a + bu$ is a unit. (Goodearl-Menal, 1988).

So we have the following implications:

Definition 3.3

A ring R is said to have unit stable range 1 if, for any $a, b \in R$, satisfying $aR + bR = R$, there exists a unit $u \in R$ such that $a + bu$ is a unit. (Goodearl-Menal, 1988).

So we have the following implications:

unit stable range 1 ⇒ stable range 1 ⇒ weak stable range 1

Definition 3.3

A ring R is said to have unit stable range 1 if, for any $a, b \in R$, satisfying $aR + bR = R$, there exists a unit $u \in R$ such that $a + bu$ is a unit. (Goodearl-Menal, 1988).

So we have the following implications:

unit stable range 1 ⇒ stable range 1 ⇒ weak stable range 1

semi strongly regular ⇒ semi unit–regular ⇒ semi one–sided unit–regular

strongly regular $\Rightarrow$unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

semi strongly regular $\not\Leftarrow$ semiregular and unit stable range 1

semi strongly regular $\not\Leftrightarrow$ semiregular and unit stable range 1

Example 3.4

semi strongly regular $\nLeftarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

◇ is semi strongly regular

semi strongly regular $\not\Leftrightarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

- ◇ is semi strongly regular
- ◇ does not have unit stable range 1

semi strongly regular $\not\Leftrightarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

- ◇ is semi strongly regular
- ◇ does not have unit stable range 1

Example 3.5

semi strongly regular $\not\Leftarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

◇ is semi strongly regular

◇ does not have unit stable range 1

Example 3.5

$R = M_2(\mathbb{Z}_2)$

◇ is semiregular

strongly regular $\Rightarrow$unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

semi strongly regular $\not\Leftarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

- $\diamond$ is semi strongly regular
- $\diamond$ does not have unit stable range 1

Example 3.5

$R = M_2(\mathbb{Z}_2)$

- $\diamond$ is semiregular
- $\diamond$ has unit stable range 1 (Chen, 2000)

semi strongly regular $\not\Leftarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

◇ is semi strongly regular

◇ does not have unit stable range 1

Example 3.5

$R = M_2(\mathbb{Z}_2)$

◇ is semiregular

◇ has unit stable range 1 (Chen, 2000)

◇ is not semi strongly regular since it has non-central idempotents

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

Definition 4.1

An element a of R is called π–regular

Definition 4.1

An element a of R is called π–regular if a power of a is regular.

Definition 4.1

An element a of R is called π–regular if a power of a is regular.
If every element of R is π–regular, then R is called π-regular (Kaplansky, 1951).

Definition 4.1

An element a of R is called π–regular if a power of a is regular.
If every element of R is π–regular, then R is called π-regular (Kaplansky, 1951).

R is π–regular $\Leftrightarrow$

Definition 4.1

An element a of R is called π–regular if a power of a is regular.
If every element of R is π–regular, then R is called π-regular (Kaplansky, 1951).

R is π–regular $\Leftrightarrow$ for any $a \in R$, there exist a positive integer n and a π–regular element $d \in R$ such that $a^n - d = 0$.

Semi π–regular ring:

G. Xiao ,W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

Semi π–regular ring:

G. Xiao ,W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

R/J is π–regular and idempotents lift modulo J.

$\Leftrightarrow$

Semi π–regular ring:

G. Xiao ,W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

R/J is π–regular and idempotents lift modulo J.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and a regular element $d \in R$ such that $a^n - d \in J$.

$\Leftrightarrow$

Semi π–regular ring:

G. Xiao ,W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

R/J is π–regular and idempotents lift modulo J.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and a regular element $d \in R$ such that $a^n - d \in J$.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and an idempotent $e \in a^n R$ such that $(1-e)a^n \in J$.

$\Leftrightarrow$

Semi π–regular ring:

G. Xiao ,W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

R/J is π–regular and idempotents lift modulo J.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and a regular element $d \in R$ such that $a^n - d \in J$.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and an idempotent $e \in a^nR$ such that $(1-e)a^n \in J$.

$\Leftrightarrow$

For any $a \in R$, there exist a positive integer n and an idempotent $e \in Ra^n$ such that $a^n(1-e) \in J$.

Semi π–regular ring:

Theorem 4.2

Semi π–regular ring:

Theorem 4.2

TFAE for a ring R:

Semi π–regular ring:

Theorem 4.2

TFAE for a ring R:

1. R/J is π–regular and idempotents lift modulo J.

Semi π–regular ring:

Theorem 4.2

TFAE for a ring R:

1. R/J is π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n and a π–regular element d of R such that $a^n - d \in J$.

Semi π–regular ring:

Theorem 4.2

TFAE for a ring R:

1. R/J is π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n and a π–regular element d of R such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer n and a π–regular element $d \in a^n R$ (resp. $d \in a^n R a^n$) such that $a^n - d \in J$.

Definition 5.1

An element a of R is called unit π–regular

Definition 5.1

An element a of R is called unit π–regular if there exists a positive integer n such that a^n is unit–regular.

Definition 5.1

An element a of R is called unit π–regular if there exists a positive integer n such that a^n is unit–regular.
R is called unit π–regular if every element of R is unit π–regular (Chen, 1998).

Semi unit π–regular ring:

Theorem 5.2

Semi unit π–regular ring:

Theorem 5.2

TFAE for a ring R:

Semi unit π–regular ring:

Theorem 5.2

TFAE for a ring R:

1. R/J is unit π–regular and idempotents lift modulo J.

Semi unit π−regular ring:

Theorem 5.2

TFAE for a ring R:

1. R/J is unit π−regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n and a unit (π-)regular element $d \in R$ (resp. $d \in a^nR$) such that $a^n - d \in J$.

Semi unit π−regular ring:

Theorem 5.2

TFAE for a ring R:

1. R/J is unit π−regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n and a unit (π-)regular element $d \in R$ (resp. $d \in a^n R$) such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^n R$, $(1-e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$.

Semi unit π–regular ring:

Theorem 5.2

TFAE for a ring R:

1. R/J is unit π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n and a unit (π-)regular element $d \in R$ (resp. $d \in a^n R$) such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^n R$, $(1-e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$.
4. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in Ra^n$, $a^n(1-e) \in J$ and $a^n b - (a^n b)^2 \in J$.

Definition 6.1

An element a of R is called strongly $\pi-$regular

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

Definition 6.1

An element a of R is called strongly $\pi-$regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly $\pi-$regular if every $a \in R$ is a strongly $\pi-$regular element, equivalently

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly π–regular rings are π–regular (Azumaya, 1954).

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly π–regular rings are π–regular (Azumaya, 1954).
Strongly π–regular rings have stable range 1 (Ara, 1996).

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly π–regular rings are π–regular (Azumaya, 1954).
Strongly π–regular rings have stable range 1 (Ara, 1996).
Strongly π–regular rings are unit π–regular.

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly π–regular rings are π–regular (Azumaya, 1954).
Strongly π–regular rings have stable range 1 (Ara, 1996).
Strongly π–regular rings are unit π–regular.

R is strongly π–regular ring ⇔

Definition 6.1

An element a of R is called strongly π–regular if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
R is called strongly π–regular if every $a \in R$ is a strongly π–regular element, equivalently R has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly π–regular rings are π–regular (Azumaya, 1954).
Strongly π–regular rings have stable range 1 (Ara, 1996).
Strongly π–regular rings are unit π–regular.

R is strongly π–regular ring ⇔ for any $a \in R$, there exists a positive integer n such that $a^n = eu = ue$ for some idempotent $e \in R$ and some unit $u \in R$ (Chin, 2004).

Semi strongly π−regular ring

Theorem 6.2

Semi strongly π–regular ring

Theorem 6.2

TFAE for a ring R:

Semi strongly π–regular ring

Theorem 6.2

TFAE for a ring R:

1. R/J is strongly π–regular and idempotents lift modulo J.

Semi strongly π–regular ring

Theorem 6.2

TFAE for a ring R:

1. R/J is strongly π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ (resp. $e \in a^n R$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$.

Semi strongly π–regular ring

Theorem 6.2

TFAE for a ring R:

1. R/J is strongly π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ (resp. $e \in a^nR$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$.
3. For any $a \in R$, there exist a positive integer n, an idempotent $e \in a^nR$ and a unit $u \in R$ such that $(1-e)a^n \in J$ and $\overline{a^n}\overline{u} = \overline{u}\overline{a^n}$ is an idempotent in R/J.

Semi strongly π–regular ring

Theorem 6.2

TFAE for a ring R:

1. R/J is strongly π–regular and idempotents lift modulo J.
2. For any $a \in R$, there exist a positive integer n, an idempotent $e \in R$ (resp. $e \in a^n R$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$.
3. For any $a \in R$, there exist a positive integer n, an idempotent $e \in a^n R$ and a unit $u \in R$ such that $(1-e)a^n \in J$ and $\overline{a^n}\,\overline{u} = \overline{u}\,\overline{a^n}$ is an idempotent in R/J.
4. For any $a \in R$, there exist a positive integer n, an idempotent $e \in Ra^n$ and a unit $u \in R$ such that $a^n(1-e) \in J$ and $\overline{a^n}\,\overline{u} = \overline{u}\,\overline{a^n}$ is an idempotent in R/J.

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

Theorem 6.3

Theorem 6.3

If, for any $a \in R$, there exist a positive integer n and strongly π–regular element d such that $a^n - d \in J$,

Theorem 6.3

If, for any $a \in R$, there exist a positive integer n and strongly π–regular element d such that $a^n - d \in J$, then R is semi strongly π–regular ring.

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

Theorem 6.3

If, for any $a \in R$, there exist a positive integer n and strongly π–regular element d such that $a^n - d \in J$, then R is semi strongly π–regular ring.

The converse of Theorem 6.3 is an open question.

Definition 7.1

An element $a \in R$ is called right weakly π–regular

Definition 7.1

An element $a \in R$ is called right weakly π–regular if there exists a positive integer n such that $a^n R = (a^n R)^2$.

Definition 7.1

An element $a \in R$ is called right weakly π–regular if there exists a positive integer n such that $a^n R = (a^n R)^2$.
R is called right weakly π–regular if any element of R is right weakly π–regular (Gupta, 1977).

Definition 7.1

An element $a \in R$ is called right weakly π–regular if there exists a positive integer n such that $a^nR = (a^nR)^2$.
R is called right weakly π–regular if any element of R is right weakly π–regular (Gupta, 1977).

R is right weakly π–regular
$\Leftrightarrow$

Definition 7.1

An element $a \in R$ is called right weakly π–regular if there exists a positive integer n such that $a^n R = (a^n R)^2$.
R is called right weakly π–regular if any element of R is right weakly π–regular (Gupta, 1977).

R is right weakly π–regular
$\Leftrightarrow$
for any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d = 0$.

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

$(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

(∗) For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.

$(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly π–regular:

$(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly π–regular:

Example: Let D be a division ring and $R = D[[x]]$.

(∗) For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly π–regular:

Example: Let D be a division ring and $R = D[[x]]$.
$\diamond$ R is local.

$(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly π–regular:

Example: Let D be a division ring and $R = D[[x]]$.

$\diamond$ R is local.

$\diamond$ R satisfies $(*)$.

$(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly π–regular:

Example: Let D be a division ring and $R = D[[x]]$.
$\diamond$ R is local.
$\diamond$ R satisfies $(*)$.
$\diamond$ R is not right weakly π–regular because $x^n \notin x^n R x^n R$ for all positive integer n.

Definition 7.2

R is called semi right weakly π–regular rings

Definition 7.2

R is called semi right weakly π–regular rings if R/J is right weakly π–regular and idempotents lift modulo J.

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

Definition 7.2

R is called semi right weakly π–regular rings if R/J is right weakly π–regular and idempotents lift modulo J.

Theorem 7.3

If R satisfies $(*)$, then R/J is a right weakly π–regular ring.

Definition 7.2

R is called semi right weakly π–regular rings if R/J is right weakly π–regular and idempotents lift modulo J.

Theorem 7.3

If R satisfies $(*)$, then R/J is a right weakly π–regular ring.

Open Question

If R satisfies $(*)$, then do idempotents lift modulo J?

($(*)$ For any $a \in R$, there exist a positive integer n and a right weakly π–regular element $d \in R$ such that $a^n - d \in J$.)

If R is right weakly π–regular, then J is nil.
(Tuganbaev, 2002)

If R is right weakly π–regular, then J is nil. (Tuganbaev, 2002)

But if R satisfies $(*)$, then J need not be nil.

If R is right weakly π–regular, then J is nil. (Tuganbaev, 2002)

But if R satisfies $(*)$, then J need not be nil.
For example, the Jacobson radical of the local ring $\mathbb{Z}_{(p)}$ is not nil.

strongly regular $\Rightarrow$ strongly π–regular $\Rightarrow$ unit π–regular $\Rightarrow$ π–regular $\Rightarrow$ weakly π–regular

(∗∗) For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

(∗∗) For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

R/J is right weakly π–regular and J is nil $\Leftrightarrow$

(∗∗) For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

R/J is right weakly π–regular and J is nil $\Leftrightarrow R$ satisfies $(**)$.

$(**)$ For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

R/J is right weakly π–regular and J is nil $\Leftrightarrow R$ satisfies $(**)$.

Definition 7.5

A ring R is called right weakly regular if $(aR)^2 = aR$ for every $a \in R$ (Ramamurthi, 1973).

(∗∗) For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

R/J is right weakly π–regular and J is nil $\Leftrightarrow R$ satisfies $(**)$.

Definition 7.5

A ring R is called right weakly regular if $(aR)^2 = aR$ for every $a \in R$ (Ramamurthi, 1973).

Proposition 7.6

R/J is right weakly regular and J is nil $\Leftrightarrow$

(∗∗) For any $a \in R$, there exist a positive integer n and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

R/J is right weakly π−regular and J is nil $\Leftrightarrow R$ satisfies $(**)$.

Definition 7.5

A ring R is called right weakly regular if $(aR)^2 = aR$ for every $a \in R$ (Ramamurthi, 1973).

Proposition 7.6

R/J is right weakly regular and J is nil $\Leftrightarrow$ for any $a \in R$, there exists $x \in RaR$ such that $a - ax \in N^*(R)$.

Theorem 8.1

TFAE for an abelian ring R:

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.
5. R is semi one–sided unit–regular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.
5. R is semi one–sided unit–regular.
6. R is semi unit π–regular.

Theorem 8.1

TFAE for an abelian ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.
5. R is semi one–sided unit–regular.
6. R is semi unit π–regular.
7. R is semi π–regular.

Theorem 8.2

TFAE for a right quasi-duo ring R:

Theorem 8.2

TFAE for a right quasi-duo ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.
5. R is semi one–sided unit–regular.
6. R is semi unit π–regular.
7. R is semi π–regular.

Theorem 8.2

TFAE for a right quasi-duo ring R:

1. R is semi strongly π–regular.
2. R is semiregular.
3. R is semi strongly regular.
4. R is semi unit–regular.
5. R is semi one–sided unit–regular.
6. R is semi unit π–regular.
7. R is semi π–regular.
8. R is semi right weakly π–regular.

Example 8.3

Let D be a simple domain that is not a division ring.

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

$\diamond$ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) \mid a, b \in D \}$$

$\diamond$ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

$\diamond$ R/J is right weakly π–regular.

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \}$$

$\diamond$ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

$\diamond$ R/J is right weakly π–regular.

$\diamond$ $J = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ is nil.

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

$\diamond$ R/J is right weakly π–regular.

$\diamond$ $J = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$ is nil.

$\diamond$ R is semi right weakly π–regular.

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

$\diamond$ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

$\diamond$ R/J is right weakly π–regular.

$\diamond$ $J = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$ is nil.

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R/J is not π–regular: For a non–zero non–unit a in D, $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$ is not π–regular.

Example 8.3

Let D be a simple domain that is not a division ring. Consider

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

◇ R is right weakly π–regular (Hong, Kim, Kwak, Lee, 2000).

◇ R/J is right weakly π–regular.

◇ $J = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$ is nil.

◇ R is semi right weakly π–regular.

◇ R/J is not π–regular: For a non–zero non–unit a in D, $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$ is not π–regular.

◇ R is not right quasi–duo (Hong, Kim, Kwak, Lee, 2000).

Exchange Rings (Warfield, 1972)

Exchange Rings (Warfield, 1972)

TFAE for a ring R:

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of R can be lifted modulo every right ideal.
(Nicholson, 1977)

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of R can be lifted modulo every right ideal.
(Nicholson, 1977)

$\diamond$ Local rings, regular, semiregular rings are exchange.

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of R can be lifted modulo every right ideal.
(Nicholson, 1977)

◇ Local rings, regular, semiregular rings are exchange.
◇ π–regular rings are exchange (Stock, 1986).

Exchange Rings (Warfield, 1972)

TFAE for a ring R:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of R can be lifted modulo every right ideal.
(Nicholson, 1977)

◇ Local rings, regular, semiregular rings are exchange.
◇ π–regular rings are exchange (Stock, 1986).
◇ Semi π–regular rings are exchange (Nicholson, 1977; Stock, 1986).

semi right weakly π-regular $\not\Rightarrow$ exchange

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

$\diamond$ R is semi right weakly π–regular.

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D \}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange:

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

◇ R is semi right weakly π–regular.

◇ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

◇ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1 - a$ are not right unit.

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange.

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1-a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1}-\overline{e}) \in (\overline{1}-\overline{k})\overline{R}$, where $\overline{k} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$.

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1-a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1} - \overline{e}) \in (\overline{1} - \overline{k})\overline{R}$, where $\overline{k} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$.

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1-a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1}-\overline{e}) \in (\overline{1}-\overline{k})\overline{R}$, where $\overline{k} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$. If $\overline{e} = \overline{0}$, then $\overline{1} \in (\overline{1}-\overline{k})\overline{R}$ so $(\overline{1}-\overline{k})\overline{R} = \overline{R}$. Let $\overline{y} \in \overline{R}$ such that $(\overline{1}-\overline{k})\overline{y} = \overline{1}$.

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

⋄ R is semi right weakly π–regular.

⋄ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

⋄ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1-a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1}-\overline{e}) \in (\overline{1}-\overline{k})\overline{R}$, where $\overline{k} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$. If $\overline{e} = \overline{0}$, then $\overline{1} \in (\overline{1}-\overline{k})\overline{R}$ so $(\overline{1}-\overline{k})\overline{R} = \overline{R}$. Let $\overline{y} \in \overline{R}$ such that $(\overline{1}-\overline{k})\overline{y} = \overline{1}$.This gives that there exists an element $b \in D$ such that $(1-a)b = 1$, which is a contradiction.

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in D\}$$

◇ R is semi right weakly π–regular.

◇ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

◇ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1} - \overline{e}) \in (\overline{1} - \overline{k})\overline{R}$, where $\overline{k} = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$. If $\overline{e} = \overline{0}$, then $\overline{1} \in (\overline{1} - \overline{k})\overline{R}$ so $(\overline{1} - \overline{k})\overline{R} = \overline{R}$. Let $\overline{y} \in \overline{R}$ such that $(\overline{1} - \overline{k})\overline{y} = \overline{1}$.This gives that there exists an element $b \in D$ such that $(1 - a)b = 1$, which is a contradiction. If $\overline{e} = \overline{1}$, then a is a right unit in a similar way.

semi right weakly π-regular $\not\Rightarrow$ exchange

Example 8.4

Let D be a simple domain that is not a division ring.

$$R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D\}$$

$\diamond$ R is semi right weakly π–regular.

$\diamond$ R is exchange iff R/J is exchange and idempotents lift modulo J (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that a and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{k}\overline{R}$ such that $(\overline{1} - \overline{e}) \in (\overline{1} - \overline{k})\overline{R}$, where $\overline{k} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$. If $\overline{e} = \overline{0}$, then $\overline{1} \in (\overline{1} - \overline{k})\overline{R}$ so $(\overline{1} - \overline{k})\overline{R} = \overline{R}$. Let $\overline{y} \in \overline{R}$ such that $(\overline{1} - \overline{k})\overline{y} = \overline{1}$.This gives that there exists an element $b \in D$ such that $(1 - a)b = 1$, which is a contradiction. If $\overline{e} = \overline{1}$, then a is a right unit in a similar way.

$\diamond$ R is not exchange.

(Ara, 1996) Strongly π–regular rings have stable range one, Amer. Math. Soc.

(Arens-Kaplansky, 1948) Topological representations of algebras, Trans. Amer. Math. Soc.

(Bass, 1964) K-Theory and stable algebra, Publ. Math.

(Chen, 1998) A note on the π–regularity of rings, Chinese Quart J. Math.

(Chen H., Chen M., 2003) On semiregular rings, New Zealand J. Math.

(Chen, 2000) Exchange rings satisfying unit 1-stable range, Kyushu J. Mathematics.

(Chin, 2004) A note on strongly π–regular rings, Acta Math. Hungarica.

(Crawley-Jonsson, 1964) Refinements for infinite direct decompositions of algebraic systems, Pacific J. Math.

(Ehrlich, 1968) Unit-regular rings, Portugal. Math.

(Goodearl, 1979) Von-Neumann Regular Rings, Pitman Publishing Limited, London.
(Goodearl-Menal, 1988) Stable range one for rings with many units, J. Pure Appl. Algebra.
(Gupta, 1977) Weakly π–regular rings and group rings, Math. J. Okayama University.
(Hong, Kim, Kwak, Lee, 2000) On weak π–regularity of rings whose prime ideals are maximal, J. Pure Appl. Algebra.
(Kaplansky, 1950) Topological representations of algebras II, Trans. Amer. Math. Soc.
(Kaplansky, 1951) Semi-simple alternative rings, Portugaliae Math.
(Li-Tong, 2002) Weak cancellation of modules and weakly stable range condition for exchange ring, Acta Math. Sinica.
(Nicholson, 1976) Semiregular rings and modules, Canadian J. Math.
(Nicholson, 1977) Lifting idempotents and exchange rings, Trans. Amer. Math. Soc.

(Nicholson-Zhou, 2005) Strong lifting, J. Algebra.
(Ramamurthi, 1973) Weakly regular rings, Canad. J. Bull.
(Stock, 1986) On rings whose projective modules have the exchange property, J. Algebra.
(Tuganbaev, 2002) Semiregular, Weakly Regular and π–regular Rings, Journal Of Mathematical Sciences.
(von Neumann, 1936) On regular rings, Proc. Nat. Acad. Sci. (USA).
(von Neumann, 1936) Examples of continuous geometries, Proc. Nat. Acad. Sci (USA).
(Warfield, 1972) Exchange rings and decomposition of modules, Math. Ann.
(Wu, 1994) Weak cancellation of modules and the weak stable range one condition, Nanjing Daxue Xuebao Shuxue Bannian Kan.
(Xiao-Tong, 2005) Generalizations of semiregular rings, Comm. Algebra.

□

God made the integers, and all the rest is the work of man

Leopold Kronecker (1823-1891)

God made the integers, and all the rest is the work of man

Leopold Kronecker (1823-1891)

Merci Beaucoup!