Skew polynomial rings and coding

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Conference Non Commutative Rings June 09

Plan

Skew polynomial rings form an interesting class of non commutative rings. We survey recent applications to coding theory

- skew cyclics codes over finite fields and Galois rings (with Boucher, Ulmer, AMC 2008)
- cyclic algebras for space time block codes (with Oggier, Belfiore, ISIT 2009)
- quasi-cyclic codes (with Yemen)
- convolutional codes (after Gluesing Luersen)

Polynomial approach to cyclic codes

$$
\begin{array}{rcl}
(\mathbb{F}_q)^n & \leftrightsquigarrow & \mathbb{F}_q[x]/(x^n-1) \\
a=(a_0,a_1,\ldots,a_{n-1}) & \leftrightsquigarrow & a(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1} \\
C & \leftrightsquigarrow & \mathcal{C}=(g \pmod{x^n-1})
\end{array}
$$

C is cyclic iff $\mathcal{C}$ is an ideal of the ring $\mathbb{F}_q[x]/(x^n-1)$
invariance by shift

$$a=(a_0,a_1,\ldots,a_{n-2},a_{n-1})\in\mathcal{C}\Rightarrow(a_{n-1},a_0,a_1,\ldots,a_{n-2})\in\mathcal{C}$$

Duality of cyclic codes

Dual Code :

$$\mathcal{C}^{\perp} = \{b \in (\mathbb{F}_q)^n \mid \forall a \in \mathcal{C}, < a, b >= 0\}.$$

$x^n - 1 \;=\; h \cdot g \;\in \mathbb{F}_q[x]$ with $h = h_0 + h_1 x + \ldots + x^k$ the check polynomial

$\Rightarrow (g)^{\perp}$ is also a cyclic code with generator the reciprocal of h the complement of g ie $h_0 x^k + h_1 x^{k-1} + \ldots + 1$

Skew polynomial rings (of automorphism type)

Let R be a ring and $\theta \in \mathrm{Aut}(R)$:

$$R[X,\theta] = \{a_0 + a_1 X + \ldots + a_n X^n \mid a_i \in R \text{ et } n \in \mathbb{N}\}.$$

1. **addition :** as in $R[X]$ componentwise
2. **multiplication :** for $a \in R$ get $X \cdot a = \theta(a) \cdot X$ and distribute ...

Example : $R = \mathbb{F}_q$ a finite field. $\Rightarrow \mathbb{F}_q[X,\theta]$ left and right euclidean

Ideals of skew polynomial rings

Two sided ideals are generated by $X^t \cdot f$ with $f \in (\mathbb{F}_q)^\theta[X^{|\theta|}]$ where $|\theta| =$ order of θ in $Gal(\mathbb{F}_q/\mathbb{F}_p)$.
Consider ideals in the quotient ring by a two sided ideal

$$\begin{array}{rcl} (\mathbb{F}_q)^n & \leftrightsquigarrow & \mathbb{F}_q[X,\theta]/(f) \\ a = (a_0, a_1, \ldots, a_{n-1}) & \leftrightsquigarrow & a(X) = a_0 + a_1 X + \ldots + a_{n-1} X^{n-1} \\ C & \leftrightsquigarrow & \mathcal{C} = (g \pmod{f}) \text{ with } f = h \cdot g \end{array}$$

$$f = X^n - 1 \Rightarrow h^\perp = \theta^k(h_0)X^k + \theta^{k-1}(h_1)X^{k-1} + \ldots + 1$$

Skew polynomial rings with coefficient ring a Galois ring

$$\varphi\colon \sum_{i=0}^{n} a_i y^i \in \mathbb{Z}_4[y] \mapsto \sum_{i=0}^{n} (a_i \bmod 2) y^i \in \mathbb{F}_2[y]$$

Definition : $GR(4^m) = \mathbb{Z}_4[y]/(h)$ with $h \in \mathbb{Z}_4[y]$ such that

1. $\varphi(h) \in \mathbb{F}_2[y]$ is unitary irreducible of degree m
2. $\xi = \tilde{y} \in \mathbb{F}_2[y]/(\varphi(h))$ generates the multiplicative group of $\mathbb{F}_{2^m}$

Representation of elements :

1. $\alpha_0 + \alpha_1\xi + \ldots + \alpha_{m-1}\xi^{m-1}$ with $\alpha_i \in \mathbb{Z}_4$
2. $a + 2b \in GR(4^m)$ with a and b in $\{0, 1, \xi, \ldots, \xi^{2^m-2}\}$

$\theta\colon a + 2b \mapsto a^2 + 2b^2$ is an automiorphism of $GR(4^m)$ of order m.
NB : $\theta(\xi) = \xi^2$.

$\boxed{\Rightarrow R[X,\theta] = GR(4^m)[X,\theta] \textbf{ is a skew polynomial rings}}$

Ideals of $GR(4^m)[X,\theta]$ and skew cyclic codes over $GR(4^m)$

Compare the situation in $\mathbb{Z}[x]$:

1. Ideals are not all principal
2. division by monic polynomials is possible.

The polynomials $f \in \mathbb{Z}_4[X^m]$ that are monic of degree n generate two sided ideals. If $n = deg(f)$ then

$$
\begin{array}{rcl}
(GR(4^m))^n & \leftrightsquigarrow & GR(4^m)[X,\theta]/(f) \\
a = (a_0, a_1, \ldots, a_{n-1}) & \leftrightsquigarrow & a(X) = a_0 + a_1 X + \ldots + a_{n-1} X^{n-1} \\
\mathcal{C} & \leftrightsquigarrow & \mathcal{C}(X) = (g \pmod{f}) \text{ avec } f = h \cdot g
\end{array}
$$

with g monic.

Self dual constacyclic codes over $GR(4^m)$

If $h\,g = X^n \pm 1$ with $h = X^k + \sum_{i=0}^{r-1} h_i X^i$, then

$$g^\perp \;=\; h_k + \theta(h_{k-1})X + \ldots + \theta^k(h_0)X^k.$$

Hence for a euclidean self dual code :

$$h = X^k + \sum_{i=1}^{k-1} \left(\theta^{k-i}(g_0^{-1})\, \theta^{k-i}(g_{k-i}) X^i \right) + \theta^r(g_0^{-1})$$

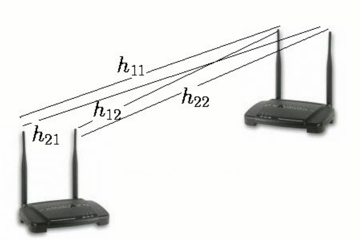
Let

$$\left(\sum_{i=0}^{k-1} g_i X^i + X^k \right) \left(\theta^k(g_0^2) + \sum_{i=1}^{k} \theta^{k-i}(g_0^2\, g_{r-i}) X^i \right) = X^{2k} \pm 1$$

for a self dual Hermitian code

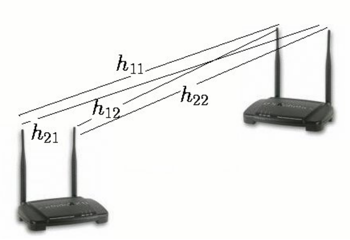
$$g^H = \sum_{i=0}^{k} \theta^{m-1+i}(h_{k-i})\, X^i$$

Space Time Codes : example



1. Time $t = 1$:
   - 1st receive antenna :$y_{11} = h_{11}x_{11} + h_{12}x_{21} + v_{11}$
   - 2nd receive antenna :$y_{21} = h_{21}x_{11} + h_{22}x_{21} + v_{21}$
2. Time $t = 2$:
   - 1st receive antenna :$y_{12} = h_{11}x_{12} + h_{12}x_{22} + v_{12}$
   - 2nd receive antenna :$y_{22} = h_{21}x_{12} + h_{22}x_{22} + v_{22}$

Space Time Codes : matrix formalism



We get the matrix equation

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \underbrace{\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}}_{\textit{space-time}\text{ codeword }\mathbf{x}} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Code design criteria (Coherent case)

- *Reliability* is modeled by the *pairwise probability of error*, bounded by

$$P(\mathbf{X} \to \hat{\mathbf{X}}) \leq \frac{const}{|\det(\mathbf{X} - \hat{\mathbf{X}})|^{2M}}.$$

- We assume the receiver knows the channel (*coherent case*).
- We need

$$\det(\mathbf{X} - \mathbf{X}') \neq 0 \quad \forall\ \mathbf{X} \neq \mathbf{X}'$$

called *fully diverse* codes.

- We attempt to maximize the *minimum determinant*

$$\min_{\mathbf{X} \neq \mathbf{X}'} |\det(\mathbf{X} - \mathbf{X}')|^2.$$

The idea behind division algebras

- The difficulty in building $\mathcal{C}$ such that

$$\det(\mathbf{X}_i - \mathbf{X}_j) \neq 0, \ \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C},$$

  comes from the *non-linearity* of the determinant.
- If $\mathcal{C}$ is taken inside an *algebra* of matrices, the problem simplifies to

$$\det(\mathbf{X}) \neq 0, \ \mathbf{0} \neq \mathbf{X} \in \mathcal{C}.$$

- A *division algebra* is a non-commutative field.

An example : cyclic division algebras

- Let $\mathbb{Q}(i) = \{a + ib,\ a, b \in \mathbb{Q}\} \supset$ information symbols.
- Let $L/\mathbb{Q}(i)$ be a *cyclic* number field of degree n.
- A *cyclic algebra* $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in L\}$$

  with basis $\{1, e, \ldots, e^{n-1}\}$ and $e^n = \gamma \in \mathbb{Q}(i)$.
- Think of $i^2 = -1$.
- A *non-commutativity rule* : $\lambda e = e\sigma(\lambda)$, $\sigma : L \to L$ the generator of the Galois group of $L/\mathbb{Q}(i)$.
- $\mathcal{A}$ is the quotient of the *skew polynomial ring* $L[e;\sigma]$ by the principal ideal $(e^n - \gamma)$.

The "Golden code"

A codeword X belonging to the Golden Code $\mathcal{G}$ has the form

$$X = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha(a+b\theta) & \alpha(c+d\theta) \\ i\bar{\alpha}(c+d\bar{\theta}) & \bar{\alpha}(a+b\bar{\theta}) \end{pmatrix}$$

where a, b, c, d are QAM symbols (that is, $a, b, c, d \in \mathbb{Z}[i]$), $\theta = \frac{1+\sqrt{5}}{2}$, $\bar{\theta} = \frac{1-\sqrt{5}}{2}$, $\alpha = 1 + i - i\theta$ and $\bar{\alpha} = 1 + i - i\bar{\theta}$. Its minimum determinant is given by

$$\delta = \min_{\mathbf{0} \neq X \in \mathcal{G}} |\det(X)|^2 = \frac{1}{5}.$$

Codes over $M_2(\mathbb{F}_2)$

When using a coset code from the "Golden code"

$$= (X_1, \dots, X_L),\ X_i \in \mathcal{G}$$

for $i = 1, \dots, L$, (Cf Construction A of Lattices from Codes)

$$\mathcal{G} = \alpha(\mathbb{Z}[i, \theta] \oplus \mathbb{Z}[i, \theta]j),$$

(where $j^2 = i$) the quotient that appears is the ring $M_2(\mathbb{F}_2)$

$$\mathcal{G}/(1+i)\mathcal{G} \simeq \mathcal{M}_2(\mathbb{F}_2),$$

A useful metric on codes over that ring to bound below the determinant is induced by the Bachoc weight defined for nonzero $M's$ by $w_B(M) = 1$ if M is invertible
$w_B(M) = 2$ if M is non-invertible

The Bachoc map

Bachoc (1997) has shown that codes over $\mathcal{M}_2(\mathbb{F}_2)$ reduce to codes over $\mathbb{F}_4 = \mathbb{F}_2(\omega)$, where $\omega^2 + \omega + 1 = 0$. Indeed, first note that

$$\mathcal{M}_2(\mathbb{F}_2) \simeq \mathbb{F}_2(\omega) + \mathbb{F}_2(\omega)j \qquad (1)$$

where $j^2 = 1$ and $j\omega = \bar{\omega}j = \omega^2 j$. The isomorphism is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto j, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \mapsto \omega.$$

Bachoc map and Ore rings

More formally denote by $\mathbb{F}_4[X;\sigma]$ the Ore ring with base field $\mathbb{F}_4$ and field automorphism $\sigma : x \mapsto x^2$. With this notation we have the ring isomorphism

$$R := \mathbb{F}_4[X;\sigma]/(X^2+1) \simeq \mathcal{M}_2(\mathbb{F}_2)$$

by identifying X and j. This isomorphism in turn induces an isomorphism of $\mathbb{F}_2$ left vector spaces

$$\phi : \mathbb{F}_4 \times \mathbb{F}_4 \to \mathcal{M}_2(\mathbb{F}_2).$$

The Bachoc map is an isometry from Bachoc weight to Hamming weight

We have that ϕ maps a pair $(a, b) \in \mathbb{F}_4 \times \mathbb{F}_4$ to a matrix in $\mathcal{M}_2(\mathbb{F}_2)$,
the elements $(a, 0)$ and $(0, b)$ can be identified with a, $bj \in R$ respectively, their image yields an invertible matrix in $\mathcal{M}_2(\mathbb{F}_2)$ whenever $a, b \in \mathbb{F}_4^*$.
These 6 elements thus correspond to the 6 invertible matrices of $\mathcal{M}_2(\mathbb{F}_2)$,
a one-to-one correspondence between elements of Hamming weight 1 in $\mathbb{F}_4^2$ and invertible matrices in $\mathcal{M}_2(\mathbb{F}_2)$.

Definitions

Linear code C of length n over a ring A : an A-submodule of A^n, i.e.,

- $x, y \in C \Rightarrow x + y \in C$;
- $\forall\lambda \in A,\ x \in C \Rightarrow \lambda x \in C$,

T : standard shift operator on A^n

$$T(a_0, a_1, \ldots, a_{n-1}) = (a_{n-1}, a_0, \ldots, a_{n-2}).$$

C quasi-cyclic of index ℓ or ℓ-quasi-cyclic : invariant under T^ℓ.
Assume : ℓ divides n
$m := n/\ell$: co-index.

Our approach

- If $\ell = 2$ and first circulant block is identity matrix, code equivalent to a so-called pure **double circulant** code.
- alternatively the generator matrix is block circulant by blocks of order ℓ
- here we view an $\ell-$ QC over A as a cyclic code of length m over A^ℓ (viewed as an $A-$module not as a ring)
- natural action of (commutative) polynomials in X with coefficients in $M_\ell(A)$
- $\Rightarrow$ How to factorize $X^m - 1$ in $M_\ell(A)[X]$?

QC codes over fields

Denote by $\mathbb{F}_{q^\ell}[X;\sigma]$ the **skew polynomial ring** ring with base field $\mathbb{F}_{q^\ell}$ and field automorphism σ;
Denote by $M_n(K)$ the ring of matrices of order n with entries in the field K.

We have the ring isomorphism

$$M_\ell(\mathbb{F}_q) \simeq \mathbb{F}_{q^\ell}[Y;\sigma]/(Y^\ell - 1)$$

which generalizes the Bachoc map

Factorization over $M_\ell(\mathbb{F}_q)[X]$

Because of the generalized Bachoc map $\mathbb{F}_{q^\ell}[X]$ is isomorphic to a subring of $M_\ell(\mathbb{F}_q)[X]$
Therefore very factorization over $\mathbb{F}_{q^\ell}[X]$ gives a factorization over $M_q^\ell(\mathbb{F}_q)[X]$.
Question When are these the only ones ?
Example : If $q = \ell = 2$ then $X^{2m} + 1 = (X^m + Y)^2$

2-QC codes over $\mathbb{F}_2$

Assume a factorization $X^n + 1 = fg$ with $f, g \in \mathbb{F}_4[X]$.
When is there a factorization $X^n + 1 = (f_1 + Yf_2)(g_1 + Yg_2)$ with $f_i, g_i \in \mathbb{F}_4[X]$ satisfying $f = f_1 + f_2$ and $g = g_1 + g_2$?
When $f \neq \sigma(f)$ we can show that there is an infinity of (explicit) solutions.
If $f = \sigma(f)$ open problem.

Cyclic Convolutional Codes (CCC)

Let A denote the auxiliary ring that enters the study of cyclic codes of length n over $\mathbb{F}$, a finite field.

$$A := \mathbb{F}[x]/(x^n - 1).$$

Let σ denote an arbitrary automorphism of A
Consider the skew polynomial ring $A[z;\sigma]$ in z with coefficients in A and the commutation rule

$$za = \sigma(a)z$$

A one sided ideal of $A[z;\sigma]$ can be regarded as a $\mathbb{F}[z]-$submodule of $\mathbb{F}[z]^n$ just by changing the order of summation between x and z. Therefore it is a convolutional code of length n over $\mathbb{F}$.

Remarks

- The case $\sigma = 1$ can be reduced to block codes.
- A can be replaced by the group algebra of an abelian group (instead of a cyclic group)
- most results are concerned with a characterization of the generator matrix
- for more info see Heide Gluesing Luerssen home page