Krull-Schmidt Theorem: the case two.

Alberto Facchini

Università di Padova, Italy

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Joint paper with Pavel Příhoda (The Krull-Schmidt Theorem in the case two, Algebr. Represent. Theory 14(3) (2011), 545–570), but also results obtained jointly with A. Amini, B. Amini, Ş. Ecevit, N. Girardi, D. Herbera and M. T. Koşan.

Theorem 1 [Krull-Schmidt-Azumaya Theorem] *Let* M *be a module that is a direct sum of modules with local endomorphism rings. Then* M *is a direct sum of indecomposable modules in an essentially unique way in the following sense. If*

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j,$$

where all the M_i*'s* $(i \in I)$ *and all the* N_j*'s* $(j \in J)$ *are indecomposable modules, then there exists a bijection* $\varphi\colon I \to J$ *such that* $M_i \cong N_{\varphi(i)}$ *for every* $i \in I$*.*

The isomorphism class of a module M_R is $\langle M_R \rangle := \{ X_R \mid X_R$ is a module and $X_R \cong M_R \}$.

Fix a class $\mathcal{C}$ of right R-modules. Assume that $\mathcal{C}$ is closed under isomorphism, direct summands and finite direct sums. Set

$$V(\mathcal{C}) := \{ \langle M_R \rangle \mid M_R \in \mathcal{C} \}.$$

($V(\mathcal{C})$ is a class, not a set in general).

Define

$$\langle M_R \rangle + \langle N_R \rangle := \langle M_R \oplus N_R \rangle$$

for every $M_R, N_R \in \mathcal{C}$. Then $V(\mathcal{C})$ becomes an additive commutative monoid.

Example:

(Krull-Schmidt-Azumaya Theorem)

$\mathcal{C} = \{ M_R \mid M_R$ is the direct sum of finitely many submodules with local endomorphism rings$\}$

$V(\mathcal{C})$ is a (possibly large) free commutative monoid:

$$V(\mathcal{C}) \cong \mathbf{N}^{(X)}.$$

In this case, the free set (class) of generators of the monoid is given by the isomorphism classes of the modules with local endomorphism ring.

What happens when the endomorphism rings End(M_i) are not local?

Let's see some special cases.

Uniserial modules

A module U_R is *uniserial* if the lattice $\mathcal{L}(U_R)$ of its submodules is linearly ordered under inclusion.

The endomorphism ring of a uniserial module has at most two maximal right (left) ideals:

Theorem 2 [F., T.A.M.S. 1996] *Let U_R be a uniserial module over a ring R, $E := \mathrm{End}(U_R)$ its endomorphism ring, $I := \{ f \in E \mid f$ is not injective$\}$ and $K := \{ f \in E \mid f$ is not surjective$\}$. Then I and K are two two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover,*

(a) *either E is a local ring with maximal ideal $I \cup K$, or*

(b) *E/I and E/K are division rings, and $E/J(E) \cong E/I \times E/K$.*

Two modules U and V are said to have

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;

2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

For instance, two injective modules have the same monogeny class if and only if they are isomorphic (Bumby's Theorem).

Theorem 3 (Weak Krull-Schmidt Theorem, [F., T.A.M.S. 1996]) *Let* $U_1, \dots,$ $U_n, V_1, \dots, V_t$ *be* $n+t$ *non-zero uniserial right modules over a ring* R*. Then the direct sums* $U_1 \oplus \cdots \oplus U_n$ *and* $V_1 \oplus \cdots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \dots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \dots, n$*.*

More generally, these results hold for modules that are *biuniform*, that is both *uniform* (of Goldie dimension 1) and *couniform* (of dual Goldie dimension 1).

Cyclically presented modules

B. Amini, A. Amini and F., *Equivalence of Diagonal Matrices over Local Rings,* J. Algebra **320** (2008), 1288–1310.

A right module over a ring R is said to be *cyclically presented* if it is isomorphic to R/aR for some $a \in R$.

If R/aR and R/bR are cyclically presented R-modules, R local, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if $\exists u, v \in U(R), r, s \in R$ with $au = rb$ and $bv = sa$.

R/aR non-zero cyclically presented $\Longrightarrow$ $\mathrm{End}_R(R/aR) \cong E/aR$, where $E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem 4 *Let a be a non-zero non-invertible element of a local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Let $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then one of the following two conditions hold:*

(a) *Either E is a local ring, or*

(b) *$E/J(E) \cong E/I \times E/K$, where E/I and E/K are division rings.*

Theorem 5

(Weak Krull-Schmidt Theorem)

Let $a_1, \dots, a_n, b_1, \dots, b_t$ *be non-invertible elements of a local ring* R*. Then*

$$R/a_1R \oplus \cdots \oplus R/a_nR \cong R/b_1R \oplus \cdots \oplus R/b_tR$$

as right R*-modules if and only if* $n = t$ *and there are two permutations* σ, τ *of* $\{1, 2, \dots, n\}$ *such that* $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ *and* $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ *for every* $i = 1, 2, \dots, n$*.*

Two $m \times n$ matrices A and B with entries in a ring R are *equivalent* matrices, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with entries in R (that is, matrices invertible in the rings $M_m(R)$ and $M_n(R)$, respectively) such that $B = PAQ$.

$\mathrm{diag}(a_1, \dots, a_n) := n \times n$ diagonal matrix

Corollary 6 *Let* $a_1, \dots, a_n, b_1, \dots, b_n \in R$, *$R$ local. Then* $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$ $\iff$ $\exists \sigma, \tau$ *with* $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ *and* $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ *for every* i*.*

Kernels of morphisms between injective indecomposable modules

We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi\colon E(A_R) \to E(B_R)$ and a homomorphism $\psi\colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

Let E_0, E_1, E'_0, E'_1 be indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_0 \to E_1, \varphi'\colon E'_0 \to E'_1$ be two non-injective morphisms. Any morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to the injective resolutions $0 \to \ker\varphi \to E_0 \xrightarrow{\varphi} E_1$ and $0 \to \ker\varphi' \to E'_0 \xrightarrow{\varphi'} E'_1$ of $\ker\varphi$ and $\ker\varphi'$. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \ker\varphi & \to & E_0 & \xrightarrow{\varphi} & E_1 \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 \\ 0 & \to & \ker\varphi' & \to & E'_0 & \xrightarrow{\varphi'} & E'_1. \end{array}$$

Notice that f_0 and f_1 are not uniquely determined by f.

Theorem 7 *E_0 and E_1 indecomposable injective right modules over a ring R, $\varphi\colon E_0 \to E_1$ a non-zero non-injective morphism, $S := \mathrm{End}_R(\ker\varphi)$, $I := \{ f \in S \mid$ the endomorphism f of $\ker\varphi$ is not a monomorphism$\}$ and $K := \{ f \in S \mid$ the endomorphism f_1 of E_1 is not a monomorphism$\} = \{ f \in S \mid \ker\varphi \subset f_0^{-1}(\ker\varphi) \}$. Then I and K are two two-sided completely prime ideals of S, and every proper right (or left) ideal of S contained either in I or in K. Moreover,*

(a) *either S is a local ring, or*

(b) *S/I and S/K are division rings and $S/J(S) \cong S/I \times S/K$.*

Theorem 8 (Weak Krull-Schmidt Theorem; F.- Ecevit-Koşan) *Let* $\varphi_i\colon E_{i,0} \to E_{i,1}$ $(i = 1, 2, \ldots, n)$ *and* $\varphi'_j\colon E'_{j,0} \to E'_{j,1}$ $(j = 1, 2, \ldots, t)$ *be non-injective morphisms between indecomposable injective modules* $E_{i,0}, E_{i,1}, E'_{j,0}, E'_{j,1}$ *over an arbitrary ring* R. *Then* $\oplus_{i=0}^{n} \ker \varphi_i \cong \oplus_{j=0}^{t} \ker \varphi'_j$ *if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \ldots, n\}$ *such that* $[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m$ *and* $[\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$ *for every* $i = 1, 2, \ldots, n$.

Couniformly presented modules

Couniform modules = of dual Godie dimension 1.

Lemma 9 *TFAE for a projective right module P_R:*

(1) P_R *is couniform.*

(2) *The module* P_R *is the projective cover of a simple module.*

(3) End(P_R) *is a local ring.*

(4) *There exists an idempotent* $e \in R$ *with* eRe *a local ring and* $P_R \cong eR$*.*

(5) P_R *is a finitely generated module with a unique maximal submodule.*

(6) Hom(P_R, R) *is a couniform module.*

We say that a module M_R is *couniformly presented* if it is non-zero and there exists an exact sequence

$$(1) \qquad 0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0$$

with both C_R and P_R couniform and P_R projective. Under these hypotheses, we will say that (1) is a *couniform presentation* of the couniformly presented module M_R.

Let M_R and M'_R be couniformly presented modules and $0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0, 0 \to C'_R \xrightarrow{\iota} P'_R \to M'_R \to 0$ two couniform presentations. Every morphism $f\colon M_R \to M'_R$ lifts to a morphism $f_0\colon P_R \to P'_R$ of the projective covers. Let $f_1\colon C_R \to C'_R$ be the restriction of f_0. We get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\ & & f_1 \downarrow & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & C'_R & \xrightarrow{\iota} & P'_R & \to & M'_R & \to & 0. \end{array}$$

Theorem 10 *Let M_R be a couniformly presented module over an arbitrary ring R, $0 \to C_R \to P_R \to M_R \to 0$ a couniform presentation of M_R, $K := \{ f \in \mathrm{End}(M_R) \mid f$ is not surjective$\}$ and $I := \{ f \in \mathrm{End}(M_R) \mid f_1 \colon C_R \to C_R$ is not surjective$\}$. Then K and I are two two-sided completely prime ideals of* $\mathrm{End}(M_R)$*, and every proper right (or left) ideal of* $\mathrm{End}(M_R)$ *is contained either in K or in I. Moreover,*

(a) *either* $\mathrm{End}(M_R)$ *is a local ring, or*

(b) $\mathrm{End}(M_R)/I$ *and* $\mathrm{End}(M_R)/K$ *are division rings, and* $\mathrm{End}(M_R)/J(\mathrm{End}(M_R)) \cong \mathrm{End}(M_R)/K \times \mathrm{End}(M_R)/I$.

Let M_R and M'_R be two couniformly presented modules and let $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$ be two couniform presentations of M,M'_R respectively. We say that M_R and M'_R *have the same lower part,* and write $[M_R]_\ell = [M'_R]_\ell$, if there exist two homomorphisms $f_0\colon P_R \to P'_R$ and $f'_0\colon P'_R \to P_R$ with $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$.

Theorem 11 (Weak Krull-Schmidt Theorem for couniformly presented modules) *Let* $M_1, \ldots, M_n, N_1, \ldots, N_t$ *be* $n+t$ *couniformly presented right* R*-modules. Then the direct sums* $M_1 \oplus \cdots \oplus M_n$ *and* $N_1 \oplus \cdots \oplus N_t$ *are isomorphic if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \ldots, n\}$ *such that* $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ *and* $[M_i]_e = [N_{\tau(i)}]_e$ *for every* $i = 1, \ldots, n$.

Local morphisms

If R and S are rings, a ring homomorphism $\varphi\colon R \to S$ is *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R.

We say that a ring R *has type* n if the factor ring $R/J(R)$ is a direct product of n division rings.

Proposition 12 *The following conditions are equivalent for a positive integer* n *and a ring* R *with Jacobson radical* $J(R)$*.*

(i) n *is the smallest integer* m *such that there exists a local homomorphism of the ring* R *into a direct product of* m *division rings.*

(ii) R *has exactly* n *distinct maximal right ideals, and they are all two-sided ideals in* R*.*

(iii) R *has exactly* n *distinct maximal left ideals, and they are all two-sided ideals in* R*.*

(iv) *The ring* R *has type* n *(i.e.,* $R/J(R)$ *is a direct product of* n *division rings).*

Moreover, if these equivalent conditions hold and, for every $i = 1, \dots, n$*,* $\varphi_i \colon R \to D_i$ *is a ring morphism of* R *into a division ring* D_i *with*

$$\varphi_1 \times \cdots \times \varphi_n \colon R \to D_1 \times \cdots \times D_n$$

a local morphism, then $\ker \varphi_1, \dots, \ker \varphi_n$ *are exactly the* n *distinct maximal right ideals and maximal left ideals of* R*.*

A ring R has type 1 if and only if it is a local ring.

We will say that a right module M_R over a ring R *has type* n if its endomorphism ring $\mathrm{End}(M_R)$ is a ring of type n.

Example 13 *(F.-Herbera)* Let E_0 and E_1 be two indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_0 \to E_1$ be a non-zero non-injective morphism. The mapping

$$\begin{aligned} \mathrm{End}_R(\ker\varphi) &\to \mathrm{End}_R(E_0)/J(\mathrm{End}_R(E_0))\times \\ &\quad \times \mathrm{End}_R(E_1)/J(\mathrm{End}_R(E_1)) \\ f &\mapsto (f_0 + J(\mathrm{End}_R(E_0)), f_1 + J(\mathrm{End}_R(E_1))) \end{aligned}$$

is a well defined local morphism. Hence $\mathrm{End}_R(\ker\varphi)$ has type ≤ 2.

General theory for modules of type 2? of type n?

$\mathcal{C} =$ a full subcategory of Mod-R.

M a non-zero right R-module, P a two-sided ideal of $\mathrm{End}_R(M)$. Let $\mathcal{P}$ be the ideal of the category $\mathcal{C}$ (the *ideal of* $\mathcal{C}$ *associated to* P) defined as follows: a morphism $f\colon X \to Y$ is in $\mathcal{P}(X,Y)$ if and only if $\beta f \alpha \in P$ for every $\alpha\colon M \to X$ and every $\beta\colon Y \to M$. If M is an object of $\mathcal{C}$, then $\mathcal{P}$ is the greatest among the ideals $\mathcal{P}'$ of $\mathcal{C}$ with $\mathcal{P}'(M,M) \subseteq P$. (In this case, as is easily seen, $\mathcal{P}(M,M) = P$.)

For every module M of type n, set $V(M_R) =$ "the set whose elements are the n ideals $\mathcal{P}_1, \dots, \mathcal{P}_n$ of the category $\mathcal{C}$ associated to the n maximal ideals $P_1, \dots, P_n$ of $\mathrm{End}_R(M)$". (The set $V(M_R)$ has cardinality exactly n.)

Theorem 14 *Let $\mathcal{C}$ be a full subcategory of* Mod-R *and $M, N \in$* Ob($\mathcal{C}$) *be R-modules of finite type. Then $M \cong N$ if and only if $V(M) = V(N)$.*

FT-R = right R-modules of finite type.

SFT-R = all right R-modules that are direct sums of finitely many right R-modules of finite type.

add($\mathcal{C}$) = closure of $\mathcal{C}$ with respect to direct summands, $\mathcal{K}$ = ideal of add($\mathcal{C}$) associated to P.

Proposition 15 *Let M be a right R-module of finite type, let P be a maximal ideal of* $\mathrm{End}_R(M)$, *let $\mathcal{P}$ denote the ideal of* SFT-R *associated to P and $\mathcal{K}$ the ideal of* add(SFT-R) *associated to P. Then* SFT-$R/\mathcal{P} \cong$ add(SFT-R)$/\mathcal{K} \cong$ mod-$\mathrm{End}_R(M)/P$.

Let $\mathcal{F}$ be the class of all canonical functors $F\colon \text{add}(\text{SFT-}R) \to \text{add}(\text{SFT-}R)/\mathcal{P}$, where $\mathcal{P} \in V(M_R)$ for some $M_R \in \text{Ob}(\text{FT-}R)$.

For every $F \in \mathcal{F}$, we can define $\dim_F(M)$ as the dimension of the vector space over $\text{End}(M)/P$ corresponding to $F(M)$.

Corollary 16 *Let M, N be objects of* add(SFT-R). *Then $M \cong N$ if and only if* $\dim_F(M) = \dim_F(N)$ *for every $F \in \mathcal{F}$.*

The Krull-Schmidt-Azumaya Theorem in the case 2.

Module of type 1 = module whose endomorphism is local.

R a fixed ring.

$\mathcal{T} = \{$ indecomposable right R-modules of type $\leq 2\}$.

Does some weak form of the Krull-Schmidt-Azumaya Theorem holds?

The situation is described by a *graph* G *associated to* R.

Vertices: the ideals $\mathcal{P}$, where $\mathcal{P}$ is the ideal of the full subcategory $\mathcal{T}$ associated to a maximal ideal P of $\mathrm{End}(M_R)$ for some $M_R \in \mathcal{T}$. $\displaystyle\bigcup_{M_R\in\mathcal{T}} V(M_R)$

Edges: the isomorphism classes $\langle M\rangle := \{\, Y \in \text{Mod-}R \mid Y \cong M \text{ in Mod-}R \,\}$ where M_R is any R-module of type 2. (For every such M_R, the endomorphism ring $\mathrm{End}(M_R)$ has exactly two maximal ideals P_1, P_2, and the edge $\langle M\rangle$ connects the vertices $\mathcal{P}_1$ and $\mathcal{P}_2$.)

Theorem 17 *For every ring R, the connected components of the graph G are either complete graphs K_α ($\alpha \geq 1$ a cardinal) or complete bipartite graphs $K_{\alpha,\beta}$ ($\alpha \geq \beta \geq 1$ cardinals).*

We can associate a commutative monoid $V(G)$ to any graph $G = (V, E)$. Given a graph $G = (V, E)$, where the elements of E are subsets of V of cardinality 2, consider the free commutative monoid $\mathbb{N}_0^{(V)}$ having as free set of generators the set of all $\delta_v \colon V \to \mathbb{N}_0$, $v \in V$, with $\delta_v(v) = 1$ and $\delta_v(w) = 0$ for every $w \in V$, $w \neq v$. If $\ell = \{v, w\} \in E$ is an edge of G, define $\delta_\ell := \delta_v + \delta_w \in \mathbb{N}_0^{(V)}$. Let $V(G)$ be the submonoid of $\mathbb{N}_0^{(V)}$ generated by (1) all the elements $\delta_\ell \in \mathbb{N}_0^{(V)}$, where ℓ ranges in E, and
(2) all the elements δ_v, where v ranges in the isolated vertices of G.

R a ring

$\mathcal{C} = \{ M_R \mid M_R$ is the direct sum of finitely many modules of type $\leq 2 \}$.

Every module in $\mathcal{C}$ has a decomposition, unique up to isomorphism, indexed in the set of all connected components of G (because if C_λ $(\lambda \in \Lambda)$ are the connected components of G, then

$$V(\mathcal{C}) = V(G) = \oplus_{\lambda \in \Lambda} V(C_\lambda),$$

so that every element of $V(\mathcal{C})$ is the sum of elements in the $V(C_\lambda)$'s in a unique way.)

Proposition 18 *Let* $M_1, \ldots, M_m, N_1, \ldots, N_n$ *be right* R*-modules of type* 2*, all in the same connected component of* G*. Assume that this connected component is a complete graph. Let* P_1, P_2 *be the two maximal ideals of* M_1*, ...,* P_{2m-1}, P_{2m} *the two maximal ideals of* M_m*,* Q_1, Q_2 *be the two maximal ideals of* N_1*, ...,* Q_{2n-1}, Q_{2n} *the two maximal ideals of* N_n*. Let* $\mathcal{P}_i$ $(i = 1, \ldots, m)$*,* $\mathcal{Q}_j$ $(j = 1, \ldots, n)$ *be the corresponding associated ideals in* Mod-R*. Then* $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n \Leftrightarrow m = n$ *and* $\exists$ *a permutation* σ *of* $\{1, 2, \ldots, 2n\}$ *such that* $\mathcal{P}_i = \mathcal{Q}_{\sigma(i)}$ *for every* $i = 1, 2, \ldots, 2n$*.*

In particular, under the hypotheses of Proposition 18, the module $M_1 \oplus \cdots \oplus M_m$ has exactly $\frac{(2m)!}{2^m \cdot m!}$ non-isomorphic direct-sum decompositions into direct sums of modules of type 2. Similarly:

Proposition 19 *$C = (V_C, E_C)$ a connected component of G, C a bipartite complete graph, $V_C = X_C \dot\cup Y_C$ the corresponding bipartition. Let $M_1, \dots, M_m$, $N_1, \dots, N_n$ be right R-modules of type 2 in the component C. It is possible to label the maximal ideals of $\mathrm{End}_R(M_i)$ and $\mathrm{End}_R(N_j)$ in such a way that the associated ideals $\mathcal{P}_1, \dots, \mathcal{P}_m, \mathcal{P}'_1, \dots, \mathcal{P}'_n$ in X corresponds to the maximal ideals $P_1, \dots, P_m, P'_1, \dots, P'_n$ of $\mathrm{End}(M_1), \dots, \mathrm{End}(M_m), \mathrm{End}(N_1), \dots, \mathrm{End}(N_n)$ respectively, and the associated ideals $\mathcal{Q}_1, \dots, \mathcal{Q}_m, \mathcal{Q}'_1, \dots, \mathcal{Q}'_n$ in Y corresponds to the maximal ideals $Q_1, \dots, Q_m, Q'_1, \dots, Q'_n$ of*

$\mathrm{End}(M_1), \ldots, \mathrm{End}(M_m), \mathrm{End}(N_1), \ldots, \mathrm{End}(N_n)$ *respectively. Then* $M_1 \oplus \cdots \oplus M_m \cong N_1 \oplus \cdots \oplus N_n$ *if and only if* $m = n$ *and there exist two permutations* σ, τ *of* $\{1, \ldots, n\}$ *such that* $\mathcal{P}_i = \mathcal{P}'_{\sigma(i)}$ *and* $\mathcal{Q}_i = \mathcal{Q}_{\tau(i)}$ *for every* $i = 1, \ldots, n$*,*

In particular, under the hypotheses of Proposition 19, the module

$$M_1 \oplus \cdots \oplus M_m$$

has exactly $n!$ direct-sum decompositions into a direct sum of modules of type 2.