On strongly prime rings, ideals and radicals

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1 Strongly prime rings

All rings in this report are associative with identity element which should be preserved by ring homomorphisms, and $R - Mod$ denotes the category of a unital left modules over the ring R. By an ideal of the ring we shall understand a two-sided ideal. $A \subset B$ means that A is proper subset of B.

Let R be a ring. The subring of $End_Z R$, generated as a ring by all left and right mutiplications l_a and r_b, where $a, b \in R$, and $l_a x = ax$, $r_b x = xb$ for $x \in R$ is called the *multiplication ring* of the ring R and will be denoted by $M(R)$. Each element of the $M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$ where $a_k, b_k \in R$. Then $\lambda x = \sum_k a_k x b_k$, $x \in R$. It's clear that the canonical embedding $R \hookrightarrow M(R)$, sending $a \in R$ to l_a is onto if and only if R is commutative. The map $\pi : M(R) \to R$, $\lambda \mapsto \lambda 1$, is an $M(R)$-module homomorphism. If R is a central simple algebra over the field F, its multiplication ring is isomorphic to $R \otimes_F R^\circ$ which is also central simple over F. If R is an Azumaya algebra then there are canonical isomorphisms $M(R) \cong R \otimes_{Z(R)} R^\circ \cong End_{Z(R)} R$.

Let M be an R-bimodule. Denote by $Z_M = Z_M(R) = \{\delta \in M \mid r\delta = \delta r, \forall r \in R\}$ the set of R-centralizing elements of the M. A bimodule M is called *centred R-bimodule* if $M = RZ_M$.

Let $\phi : R \to S$ be a ring homomorphism. Then S becomes a canonical R-bimodule. We call ϕ a *centred homomorphism* if S is a centred R-bimodule under this structure. And then the ring S is called *centred extension* of the ring R (via ϕ). Of course, $Z_S = Z_S(R)$ is a subring of the ring S. It easily follows from the definition that each centred extension of the ring R can be obtained as a factor ring of a semigroup ring $R[G]$ where G is a free semigroup with unit. Rings and their centred homomophisms form a category which is called Procesi category. For a semiprime ring R we denote $Q(R)$ the central closure and by$F(R)$ the extended centroid of the ring R. By definition, $F(R)$ is the centre of $Q(R)$ and is a field when R is a prime ring. See [3],[12] for definitions and basic properties of these rings.

Let M be a nonzero left R-module. M is called *strongly prime* if a for any non-zero $x, y \in M$, there exits finite set of elements $\{a_1, ..., a_n\} \subseteq R, \quad n = n(x, y)$, such that $Ann_R\{a_1x, ..., a_nx\} \subseteq Ann_R\{y\}$ (see [2]). Other equivalent definitions and some properties of the strongly prime modules can also be found in [2], [12]. Taking $M = R$ in the definition of the strongly prime module over R, the notion of left strongly prime ring is obtained (see [6]).

When we look at a ring R as an R-bimodule, this means that we consider R as the left $M(R)$-module. Ring R is called *strongly prime* if R is a strongly prime module over its multiplication ring. We call an element $a \in R$ a *symmetric zero divisor* if for any finite subset of elements $\{a_1, ..., a_n\} \subseteq (a)$, $Ann_{M(R)}\{a_1, ..., a_n\} \nsubseteq Ann_{M(R)}\{1_R\}$. Of course, when R is commutative, taking $n = 1$ and$a_1 = a$, we obtain the usual definition of zero divisors. Denote $zd(R)$ the set of zero divisors of the ring R.

Now we give the main characterizations of a strongly prime rings.

Theorem 1.1. *For any nonzero ring R the following conditions are equivalent:*

(1) R is a strongly prime ring;

(2) $zd(R) = 0$;

(3) R is a prime ring and the central closure $Q(R)$ *of the ring R is a simple ring;*

(4) for any nonzero $a, b \in R$, *there exist* $\lambda_1, ..., \lambda_n \in M(R)$ *such that* $Ann_{M(R)}\{\lambda_1 a, ..., \lambda_n a\} \subseteq Ann_{M(R)}\{b\}$;

(5) for any nonzero $a \in R$, there exist $\lambda_1, ..., \lambda_n \in M(R)$ such that $Ann_{M(R)}\{\lambda_1 a, ..., \lambda_n a\} \subseteq Ann_{M(R)}\{1_R\}$;

(5′) *for any nonzero $a \in R$, there exist $a_1, ..., a_n \in (a)$, such that*

$$\sum_i x_i a_k y_i = 0, \text{ for all } 1 \leq k \leq n, \text{ implies } \sum_i x_i y_i = 0;$$

(6) there exists a centred monomorphism $\phi : R \to K$ where K is a simple ring;

(7) there exists a centred monomorphism $\phi : R \to S$, where the ring S has the following property: for each nonzero ideal $I \subseteq R$, its extension I^ε in S, $I^\varepsilon = SIS$, is equal to S.

Particularly, by (2) of this theorem, each ring which is not strongly prime has nonzero symmetric zero divisors. It is also clear that a strongly prime ring is left and right strongly prime in the sense of Handelman-Lawrence. We note that for any strongly prime ring the central closure coincides with the right and left Martindale's quotient rings, and so with the symmetric ring of quotients.

The central closure $Q(R)$ of any strongly prime ring R has an important universal property. In [8] the following result was proved. Let the ring$R$ be centrally embedded into a ring S, in which for each nonzero ideal $I \subseteq R$, its extension I^ε in S,$I^\varepsilon = SIS$ is equal to S. Then R is strongly prime and there exists a unique centred homomorphism $\rho : Q(R) \to S$, extending the given embedding, and sending the extended centroid $F = Z(Q(R))$ of the ring$R$ into $Z(S)$ (see [8], Theorems 2 and 5). This generalizes Amitsur's result, proved for simple rings S (see [1], Theorem 18). Particularly this universal property shows that the simple ring $Q(R)$ is a minimal centred extension satisfying (7) of the Theorem 1.1.

Theorem 1.2. *A ring R is strongly prime if and only if its multiplication ring $M(R)$ is strongly prime.*

In this case their extended centroids are canonically isomorphic, and the central closure $Q(M(R)) \cong Q \otimes_F Q^\circ$, where $Q = Q(R)$.

Theorem 1.3. *Let R be a strongly prime ring. If a ring S is Morita equivalent to the ring R, then S is strongly prime and extended centroids of R and S are isomorphic.*

We recall that a ring R is *semiprime* if it does not contain any nonzero nilpotent ideals. It is well known that in a semiprime ring left and right annihilators of an ideal coincide, so we can speak about ideals with zero annihilators. It is also clear that an ideal of a semiprime ring R isessential as an $M(R)$-submodule if and only if it has a zero annihilator. A finite set $A = \{a_1, ..., a_n\} \subseteq R$ is called an *insulator*, if

$$Ann_{M(R)}\{a_1, ..., a_n\} \subseteq Ann_{M(R)}\{1_R\};$$

i.e. if $\lambda a_1 = ... = \lambda a_n = 0$, implies $\lambda 1 = 0$.The set $In(R)$ of the insulators of a ring R is evidently closed under multiplication. In a semiprime ring R, insulators can be characterised in terms of the central closure $Q(R)$ and extended centroid $F(R)$ of the ring. Indeed, using Theorem 32.3 in [12], we obtain the following

Proposition 1.4. *In any semiprime ring R, a finite subset $A = \{a_1, ..., a_n\}$ is aninsulator if and only if* $1 \in AF$, *i.e. if*

$$a_1\varphi_1 + ... + a_n\varphi_n = 1,$$

with suitable φ_k from the extended centroid F of R.

Let R be a ring. Denote by $\mathcal{F}$ the set of right ideals in R containing an insulator. Analogously we define the set $\mathcal{F}'$ as the left ideals of R containing an insulator. If R is commutative, any ideal generated by elements of an insulator is dense. It will follow from the proof of Proposition 1.5 below that in any commutative ring $\mathcal{F}$ is a Gabriel filter. We remind that a semiprime ring R is called strongly semiprime if for each essential ideal I of the ring, $R \in \sigma_{M(R)}[I]$ (see [12]). It easily follows from the definitions that R is strongly semiprime if and only if each essential ideal contains an insulator. Clearly each strongly prime ring is strongly semiprime.

Proposition 1.5. *If R is a strongly semiprime ring, then $\mathcal{F}$ and $\mathcal{F}'$ are symmetric Gabriel filters. Corresponding left and right localizations form a biradical in the sense of Jategaonkar i.e. corresponding torsion submodules inR/A coincide for each ideal $A \subseteq R$.*

Theorem 1.6. *Let R be a strongly semiprime ring. Then the canonical map*

$$\phi : Q(R) \otimes_R Q(R) \to Q(R)$$

is an isomorphism, and $Q(R)$ is flat as a left and a right as R-module.

A ring homomorphism, $\phi : R \to S$, for which the canonical map $S \otimes_R S \to S$ is an isomorphism and which induces the structure of a right (left) flat R-module is called a right (left) flat epimorphism. The proved theorem means that for a strongly semiprime ring, canonical embedding $R \to Q(R)$ is right and left flat epimorphism. By a theorem of Popescu-Spircu, for each right flat epimorphism $\phi : R \to S$, the set of right ideals

$$\mathcal{F} = \{U_R \subseteq R \mid \phi(U)S = S\}$$

is a Gabriel filter and S is canonically isomorphic to the quotient ring $Q_{\mathcal{F}}(R)$. It is well known that for any right flat epimorphism $\phi : R \to S$, $M \otimes_R S \cong Q_{\mathcal{F}}(M)$, for each$M \in Mod - R$, i.e. the localization, associated with a flat epimorphism, is perfect.

Applying Popescu-Spircu Theorem to the embedding $R \to Q(R)$ for a strongly semiprime ring R and the characterisation of Gabriel filters in Proposition 1.5 and Theorem 1.6 we obtain the following:

Theorem 1.7. *Let R be a strongly semiprime ring. Then $Q(R) \otimes_R Q(R) \cong Q(R)$, $Q(R)$ is flat as left and right R-module. Thes ets $\mathcal{F}$ and $\mathcal{F}'$ are symmetric Gabriel filters, the corresponding localizations are perfect, and the central closure $Q(R)$ is canonically isomorphic to the guotient ring of R with respect to$\mathcal{F}$ and $\mathcal{F}'$.*

It is worth noting, that the following lemma implies one of the equivalent conditions of the theorem Popescu-Spircu and from it we could regain all the statements of the Theorem 1.7.

Lemma 1. *Let R be a strongly semiprime ring. Then for every $q \in Q(R)$ there exist elements $i_1, ..., i_n \in R$ and $\psi_1, ..., \psi_n \in F$, such that qi_k, $i_k q \in R$, and $\sum_k i_k \psi_k = 1$.*

2 Strongly prime ideals

An ideal $\mathfrak{p} \subset R$ is called *strongly prime* if the factor ring $R/\mathfrak{p}$ is a strongly prime ring. We can adapt the Theorem 1.1 for equivalent characterizations of th estrongly prime ideal. From the (5) of this theorem we obtain the following:

Proposition 2.1. *An ideal* $\mathfrak{p} \subset R$ *is strongly prime if and only if for each* $a \notin \mathfrak{p}$, *there exist elements* $a_1, ..., a_n \in (a)$, $n = n(a)$, *such that for each* $\lambda \in M(R)$ *with* $\lambda 1 \notin \mathfrak{p}$, *at least one of elements* $\lambda a_k \notin \mathfrak{p}$.

Clearly, maximal ideals are strongly prime. It is well known that in PI rings each prime ideal is strongly prime. Of course, any strongly prime ideal is prime by (3) of Theorem 1.1. Since not each prime ring has a simple central closure, prime ideals are not necessarily strongly prime. Using standard arguments we easily obtain from Theorem 1.3, that strongly prime ideals are preserved under Morita equivalences. If $\phi : R \to S$ is a centred homomorphism of rings, and $\mathfrak{q} \subset S$ is a strongly prime ideal, we easily obtain from(6) of Theorem 1.1 that $\mathfrak{p} = \phi^{-1}\mathfrak{q}$ is a strongly prime ideal in R. The intersection of all strongly prime ideals of the ring R we call a *strongly prime radical* and denote it by $sr(R)$. We give a characterisation of the strongly prime radical of the ring. Let $R[X_1, ...X_n]$ be a polynomial ring over the ring R with commuting or noncommuting indeterminates.

Theorem 2.2. $a \in sr(R)$ *if and only if for each* $n \in I\!N$ *and arbitrary elements* $a_1, ..., a_n \in (a)$, *the ideal in* $R[X_1, ..., X_n]$, *generated by the polynomial* $a_1X_1 + ... + a_nX_n - 1$ *contains* 1.

Proof. If some polynomial $a_1X_1 + ... + a_nX_n - 1$ generates a proper ideal in $R[X_1, ..., X_n]$, we can take a maximal ideal $\mathcal{M} \subset R[X_1, ..., X_n]$ containing this polynomial. Evidently $a \notin \mathcal{M}$.So we have the centred homomorphims $\phi : R \to R[X_1, ..., X_n]/\mathcal{M}$ with $\phi a \neq 0$ and $\phi^{-1}\mathcal{M}$ is a strongly prime ideal in R not containing a. This implies $a \notin sr(R)$.Now assume $a \notin sr(R)$. Then $a \notin \mathfrak{p}$ for some strongly prime ideal $\mathfrak{p} \subset R$ and therefore $(\bar{a})^{\varepsilon} = Q(R/\mathfrak{p})$ yielding an expression

$$\bar{a}_1u_1 + ... + \bar{a}_nu_n = 1 \text{ in } Q(R/\mathfrak{p}), \text{ with } \bar{a}_1, ..., \bar{a}_n \in (\bar{a}), \quad u_1, ..., u_n \in F(R/\mathfrak{p}).$$

So the polynomial $a_1X_1 + ... + a_nX_n - 1$ is in the kernel of the homomorphismfrom $R[X_1, ..., X_n]$ to $Q(R/\mathfrak{p})$, which sends X_k to the u_k, $1 \leq k \leq n$. Thus the ideal generated by this polynomial is proper.

□

This theorem is an analogue of the well known fact that an element a ofthe commutative ring R is nilpotent if and only if the polynomial $aX - 1$ is invertible in $R[X]$. Since each maximal ideal is strongly prime, the strongly prime radical of the ring is contained in the Brown-McCoy radical.

Theorem 2.3. *Strongly prime radical $sr(R)$ of the nonzero ring contains the Levitzki radical L(R).*

Proof. We recall that the Levitzki radical is the largest locally nilpotent ideal of the ring. If some element $a \in L(R)$ is not in the strongly prime radical, we would have an expression $(*) \quad \bar{a}_1 u_1 + ... + \bar{a}_n u_n = 1 \ \text{in} Q(R/\mathfrak{p})$, with $a_1, ..., a_n \in (a), \ \ u_1, ..., u_n \in F(R/\mathfrak{p})$, for some strongly prime ideal $\mathfrak{p} \subset R$. Because set $A = \{a_1, ..., a_n\}$ is in $L(R)$, there exists $m \in I\!N$ such that all products $a_{k_1}...a_{k_m}$ with $a_{k_l} \in A$ are zero. Then the m-th power of the expression $(*)$ would give a contradiction.

□

It would be interesting to know if or under which conditions the upper nilradical of the ring is contained in $sr(R)$.

Recall that a non-empty subset $A \subseteq R$ of a ring is an *m-system* if $1 \in A$ and for each $a, b \in A$, $arb \in A$ for some $r \in R$. Two main properties of the m-systems are well known: a complement of a prime ideal is an m-system, and each ideal maximal with respect to being disjoint with A is prime. Now we introduce the notion of a strongly multiplicative set of a ring and characterize strongly prime ideals in terms of these sets. We call a subset $\mathcal{S} \subseteq R$ *strongly multiplicative*, or *sm-set*, if $1 \in \mathcal{S}$ and for any $a \in \mathcal{S}$ there exist elements $a_1, ..., a_n \in (a), \ (n = n(a))$, sucht hat for each $\lambda \in M(R)$ with $\lambda 1 \in \mathcal{S}$, we have $\lambda a_k \in \mathcal{S}$ for some $1 \leq k \leq n$.

Proposition 2.4. *If $\mathfrak{p} \subset R$ is a strongly prime ideal, its complement is a strongly multiplicative set.*

Indeed, this Proposition is just another form of Proposition2.1. Other examples of sm-sets are related to any ideal $I \subset R$. The set $\mathcal{S} = \{1+i, \ i \in I\}$ is an sm-set: for each $a = 1+i, \ i \in I$ take $n = 1, \ a_1 = a$. If $\lambda 1 = 1+j, \ j \in I$, then $\lambda a = 1 + j + \lambda i \in \mathcal{S}$, showing that $\mathcal{S}$ is strongly multiplicative.

Theorem 2.5. *Let $\mathcal{S} \subset R, \ 0 \notin \mathcal{S}$ be a strongly multiplicative set. Each ideal $\mathfrak{p} \subset R$, maximal with respect to $\mathfrak{p} \cap \mathcal{S} = \emptyset$, is strongly prime.*

Proof. Let $x \notin \mathfrak{p}$. Then $p + \mu_0 x = a \in \mathcal{S}$, for some $p \in \mathfrak{p}$ and $\mu_0 \in M(R)$. Let $a_k = \lambda_k a = \lambda_k p + \lambda_k \mu_0 x \in (a), \ 1 \leq k \leq n$ be elements corresponding to a in definition of the sm-sets. Let $\lambda 1 \notin \mathfrak{p}$. Then $q + \nu_0 \lambda 1 = (l_q + \nu_0 \lambda) 1 = \lambda' 1 \in \mathcal{S}$, where $l_q \in M(R)$ is the left multiplication by q. Then for some k, $\lambda' a_k \in \mathcal{S}$ thus not in $\mathfrak{p}$. So we have

$$\lambda' a_k = (l_q + \nu_0 \lambda)(\lambda_k p + \lambda_k \mu_0 x) = q a_k + \nu_0 \lambda \lambda_k p + \nu_0 \lambda \lambda_k \mu_0 x \notin \mathfrak{p}.$$

But $q a_k$ and $\nu_0 \lambda \lambda_k p$ are in $\mathfrak{p}$, so $\lambda \lambda_k \mu_0 x \notin \mathfrak{p}$. Thus, for each $x \notin \mathfrak{p}$, there exist a finite set of elements $x_k = \lambda_k \mu_0 x \in (x)$, such that for each $\lambda \in M(R)$ with $\lambda 1 \notin \mathfrak{p}$, at least one of the elements $\lambda x_k \notin \mathfrak{p}$. B yProposition 2.1, the ideal $\mathfrak{p}$ is strongly prime.

□

Let $\mathcal{S} \subset R$ be a strongly multiplicative set. Similarly to the commutative case, we define the set $\mathcal{S}' = \{u \in R \mid (u) \cap \mathcal{S} \neq \emptyset\}$ and call it the *saturation* of $\mathcal{S}$. If $\mathcal{S}' = \mathcal{S}$, we call $\mathcal{S}$ *saturated*. Denote by H the union of all strongly prime ideals $\mathfrak{p} \subset R$ disjoint with R. We have shown that $H \neq \emptyset$ when $0 \notin \mathcal{S}$.

Proposition 2.6. *Let $\mathcal{S}$ be a strongly multiplicative set. Then $\mathcal{S}'$ is also strongly multiplicative and $\mathcal{S}' = R \setminus H$ - the complement to the union of all strongly prime ideals disjoint with $\mathcal{S}$.*

The proof is analogous to the commutative case.

Corollary 2.7. *For a commutative ring saturated strongly multiplicative sets are the usual multiplicative sets.*

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