Right Gaussian Rings and Skew Power Series Rings

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Based on a joint work with R. Mazurek

Theorem 1 (J. Brewer, E. Rutter and J. Watkins, (1977))

For any commutative power series ring $R[[x]]$ the following conditions are equivalent:

(1) $R[[x]]$ *is Bézout*

(2) $R[[x]]$ *has weak dimension less or equal to one*

(3) R *is* $\aleph_0$*-injective von Neumann regular*

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Theorem 2 (D. Herbera, (2003))

Let R be a strongly regular ring. The following conditions are equivalent:

(1) $R[[x]]$ *is right Bézout.*

(2) $R[[x]]$ *is Bézout.*

(3) $R[[x]]$ *has weak dimension less or equal to one.*

(4) R *is* $\aleph_0$*-injective.*

Theorem 3 (A. Tuganbaev, (1990))

Let σ be an injective endomorphism of a ring R. Then the following conditions are equivalent:

(1) $R[[x;\sigma]]$ *is right Bézout and* R *is semicommutative.*

(2) $R[[x;\sigma]]$ *is right Bézout and* R *is right quasi-duo.*

(3) $R[[x;\sigma]]$ *is right distributive.*

(4) R *is* $\aleph_0$*-injective strongly regular,* σ *is bijective and* $\sigma(e) = e$ *for any idempotent* $e = e^2 \in R$*.*

Theorem 3 (A. Tuganbaev, (1990))

Let σ be an injective endomorphism of a ring R. Then the following conditions are equivalent:

1. $R[[x;\sigma]]$ *is right Bézout and* R *is semicommutative.*
2. $R[[x;\sigma]]$ *is right Bézout and* R *is right quasi-duo.*
3. $R[[x;\sigma]]$ *is right distributive.*
4. R *is* $\aleph_0$*-injective strongly regular,* σ *is bijective and* $\sigma(e) = e$ *for any idempotent* $e = e^2 \in R$.

Theorem 4 (A. Tuganbaev, (1987))

Let R be an abelian ring and σ an automorphism of R such that $\sigma(e) = e$ for any idempotent $e = e^2 \in R$. Then the following conditions are equivalent:

1. $R[[x;\sigma]]$ *has weak dimension less or equal to one.*
2. *All 2-generated right ideals of* $R[[x;\sigma]]$ *are flat.*
3. R *is* $\aleph_0$*-injective strongly regular.*

Theorem 5 (R. Mazurek, M.Z., (2009))

Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:

(1) $R[[x;\sigma]]$ *is right distributive and* σ *is injective.*

(2) $R[[x;\sigma]]$ *is right distributive and right duo.*

(3) $R[[x;\sigma]]$ *is right Bézout and right quasi-duo.*

(4) $R[[x;\sigma]]$ *is right Bézout and abelian, and* σ *is injective.*

(5) $R[[x;\sigma]]$ *is right Bézout and semicommutative, and* σ *is injective.*

(6) *All 2-generated right ideals of* $R[[x;\sigma]]$ *are flat,* R *is abelian,* σ *is bijective and* $\sigma(e) = e$ *for any* $e = e^2 \in R$.

(7) $R[[x;\sigma]]$ *has weak dimension less or equal to one and* $R[[x;\sigma]]$ *is right duo.*

(8) R *is* $\aleph_0$*-injective strongly regular,* σ *is bijective and* $\sigma(e) = e$ *for any* $e = e^2 \in R$.

For a commutative ring R and $f \in R[x]$, $c(f)$ denotes the ideal of R generated by coefficients of f. R is called Gaussian ring if $c(fg) = c(f)c(g)$ for all $f, g \in R[x]$.

Theorem 6 (D.D. Anderson, V. Camillo, (1998))

For a commutative ring R, the following conditions are equivaent:

(1) $R[[x]]$ *is Gaussian*

(2) $R[[x]]$ *is distributive*

(3) $R[[x]]$ *has weak dimension less or equal to one*

(4) $R[[x]]$ *is Bézout*

(5) R *is $\aleph_0$-injective von Neumann regular*

For a ring R and a polynomial $f \in R[x]$, let $c_r(f)$ denote the right ideal of R generated by the coefficients of f. Obviously, for any $f, g \in R[x]$ we have $c_r(fg) \subseteq c_r(f)c_r(g)$.

Definition 7

A ring R is *right Gaussian* if $c_r(fg) = c_r(f)c_r(g)$ for any $f, g \in R[x]$.

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

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- *If a ring R is right Gaussian, then R is right duo.*
- *If a ring R is right Gaussian, then so is any homomorphic image of R*
- *A direct product ring $\prod_{i \in I} R_i$ is right Gaussian if and only if each component ring R_i is.*

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- *If a ring R is right Gaussian, then so is any homomorphic image of R*
- *A direct product ring $\prod_{i \in I} R_i$ is right Gaussian if and only if each component ring R_i is.*
- *A ring R is right Gaussian if and only if R is right duo and every homomorphic image of R is Armendariz.*

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Theorem 8

Let R be a right Gaussian ring, P an ideal of R such that $S = R \setminus P$ is a right denominator set in R, and R_S a right ring of quotients with respect to S. Then the following conditions are equivalent:

(1) *R_S is right Gaussian.*

(2) *R_S is right duo.*

(3) *For any $a \in R$ we have $Sa \subseteq aS$ or $as = 0$ for some $s \in S$.*

Theorem 5 (R. Mazurek, M.Z., (2009))

Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:

(1) $R[[x;\sigma]]$ *is right distributive and* σ *is injective.*

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(3) $R[[x;\sigma]]$ *is right Bézout and right quasi-duo.*

(4) $R[[x;\sigma]]$ *is right Bézout and abelian, and* σ *is injective.*

(5) $R[[x;\sigma]]$ *is right Bézout and semicommutative, and* σ *is injective.*

(6) *All 2-generated right ideals of* $R[[x;\sigma]]$ *are flat,* R *is abelian,* σ *is bijective and* $\sigma(e) = e$ *for any* $e = e^2 \in R$.

(7) $R[[x;\sigma]]$ *has weak dimension less or equal to one and* $R[[x;\sigma]]$ *is right duo.*

(8) R *is* $\aleph_0$*-injective strongly regular,* σ *is bijective and* $\sigma(e) = e$ *for any* $e = e^2 \in R$.

Using totally different arguments than in commutative case we can prove the following:

Theorem 9 (R. Mazurek, M.Z., (2011))

Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:

(1) $R[[x;\sigma]]$ *is right Gaussian.*

(2) $R[[x;\sigma]]$ *is right distributive and σ is injective.*

(3) R *is $\aleph_0$-injective strongly regular, and σ is bijective and $\sigma(e) = e$ for any idempotent $e = e^2 \in R$.*

- Let R be a commutative ring, and denote by $Q(R)$, the total ring of quotients of R. An ideal I of R, is invertible if $I \cdot I^{-1} = R$, where $I^{-1} = \{r \in Q(R) : rI \subseteq R\}$.

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- Recall that a commutative ring R is a Prüfer ring (if R is domain, then R is called Prüfer domain) if every finitely generated regular ideal of R is invertible.

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- Recall that a commutative ring R is a Prüfer ring (if R is domain, then R is called Prüfer domain) if every finitely generated regular ideal of R is invertible.
- For a commutative domain R we have

$$R \text{ is semihereditary } \Leftrightarrow$$

$$\Leftrightarrow R \text{ has weak dimension less or equal to one } \Leftrightarrow$$

$$\Leftrightarrow R \text{ is distributive } \Leftrightarrow R \text{ is Gaussian } \Leftrightarrow R \text{ is Prüfer}$$

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- Is it true that if R is right distributive ring, then R is right Gaussian?

Answer: NO!

Theorem 10 (R.Mazurek, M.Z. (2011))

If R is a right duo right distributive ring, then R is right Gaussian.

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- If for any $s, t, v \in S$, $s < t$ implies $sv < tv$ and $vs < vt$, then $(S, \leq)$ is said to be *a strictly ordered monoid.*

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- A subset $T \subseteq S$ is *artinian* if every strictly decreasing sequence of elements of S is finite.
- A subset $T \subseteq S$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite.

Thus a subset $T \subseteq S$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements.

For given a ring R and a strictly ordered monoid $(S, \leq)$, consider the set A of all formal series $f = \sum_{s \in S} a_s s$, where $s \in S$ and $a_s \in R$, whose support $supp(f) = \{s \in S : a_s \neq 0\}$ is artinian and narrow.

If $f, g \in A$ and $s \in S$, it turns out that the set

$$X_s(f,g) = \{(x,y) \in supp(f) \times supp(g) : s = xy\}$$

is finite and the set $T = \{xy : x \in supp(f), y \in supp(g)\}$ is artinian and narrow. Thus one can define the product fg of $f = \sum_{s \in S} a_s s, g = \sum_{t \in S} b_t t \in A$ as follows:

$$fg = \sum_{v \in S} (\sum_{s,t \in S, st = v} a_s b_t) v$$

(by convention, a sum over the empty set is 0).

With pointwise addition and multiplication as defined above, A becomes a ring, called the ring of generalized power series with coeficients in R and exponents in S, and is denoted by $R[[S]]$.

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- monoid rings - a monoid S with trivial $\leq$
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- power series rings - $S = \mathbb{N} \cup \{0\}$ with usual addition, and usual $\leq$
- Laurent series rings - $S = \mathbb{Z}$ with usual addition and usual $\leq$
- the Malcev - Neumann construction - $(S, \leq)$ a totally ordered group

Theorem 11

Let R be a ring, and $(S, \leq)$ a nontrivial positively strictly ordered monoid. Then the following conditions are equivalent:

(1) *$R[[S]]$ is a right Gaussian ring and S is right chain monoid.*

(2) *$R[[S]]$ is right duo right distributive.*

(3) *Either*

(a) *S is cyclic and R is $\aleph_0$-injective strongly regular*

or

(b) *S is not cyclic, S is a right chain monoid and R is a finite direct product of division rings.*

THANK YOU FOR YOUR ATTENTION.