Euclidean pairs, Quasi Euclidean rings and Continuant Polynomials

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In all the talk $R$ will stand for a unital associative ring.

1 The Euclidean pair $(a, b)$ and its associated continuant polynomials.

Definitions 1.1. (a) An ordered pair $(a, b) \in R^2$ is a right Euclidean pair if there exist elements $(q_1, r_1), \ldots, (q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geq 0$) such that $a = bq_1 + r_1$, $b = r_1q_2 + r_2$, and

\[ r_{i-1} = r_i q_{i+1} + r_{i+1} \quad \text{for } 1 < i \leq n, \quad \text{with } r_{n+1} = 0. \]

The notion of a left Euclidean pair is defined similarly.

(b) A ring $R$ is right quasi Euclidean if every pair $(a, b) \in R^2$ is right Euclidean.

(c) Let $T = \{t_1, t_2, \ldots \}$ be a countable set of noncommuting variables, and let $\mathbb{Z}\langle T \rangle$ be the free $\mathbb{Z}$-algebra generated by $T$. We define the $n$-th right continuant polynomials

\[ p_n(t_1, \ldots, t_n) \in \mathbb{Z}\langle t_1, \ldots, t_n \rangle \subseteq \mathbb{Z}\langle T \rangle \]

by $p_0 = 1$, $p_1(t_1) = t_1$, and inductively for $i \geq 2$ by

\[ p_i(t_1, \ldots, t_i) = p_{i-1}(t_1, \ldots, t_{i-1}) t_i + p_{i-2}(t_1, \ldots, t_{i-2}). \]

Thus, $p_2(t_1, t_2) = t_1t_2 + 1$, $p_3(t_1, t_2, t_3) = t_1t_2t_3 + t_3 + t_1$, etc.

Notation: $P(q) = \begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$

*Are there connections between these three notions?*

Let us consider an easy example:

$(a, b) = (22, 8) \in \mathbb{Z}^2$ we write

$22 = 8 \times 2 + 6$

$8 = 6 \times 1 + 2$

$6 = 2 \times 3$

we then have:

$(22, 8) = (8, 6) \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$
\[(22, 8) = (6, 2) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}
\]
\[(22, 8) = (2, 0) \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}
\]

In general for a Euclidean pair \((a, b)\) \(a = bq_1 + r_1, \ b = r_1 q_2 + r_2,\) and

\((*\) \(r_{i-1} = r_i q_{i+1} + r_{i+1}\) for \(1 < i \leq n,\) with \(r_{n+1} = 0.\)

We will get that

\[(a, b) = (r_n, 0)P(q_{n+1})P(q_n)\cdots P(q_1)
\]

Now, looking at the product \(P(t_1)P(t_2)\cdots P(t_n)\) we have

\[P(t_1)P(t_2) = \begin{pmatrix} t_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t_1 t_2 + 1 & t_1 \\ t_2 & 1 \end{pmatrix}
\]

and

\[P(t_1)P(t_2)P(t_3) = \begin{pmatrix} t_1 t_2 + 1 & t_1 \\ t_2 & 1 \end{pmatrix} \begin{pmatrix} t_3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t_1 t_2 t_3 + t_1 + t_3 & t_1 t_2 + 1 \\ t_2 t_3 + 1 & t_2 \end{pmatrix}
\]

In general:

\[P(t_1)P(t_2)\cdots P(t_n) = \begin{pmatrix} p_n(t_1, \ldots, t_n) & p_{n-1}(t_1, \ldots, t_{n-1}) \\ p_{n-2}(t_2, \ldots, t_n) & p_{n-2}(t_2, \ldots, t_{n-1}) \end{pmatrix}
\]

**Examples 1.2.**

1. \((bq, b), (a, 0)\) are Euclidean pairs for any \(a, b, q \in R.\)

2. If \((a, b)\) is a Euclidean pair and \(c \in R\) then \((b, a), (ca, cb), (ac+b, a), (bc + a, b)\) are Euclidean pairs.

3. If \(a, b \in R\) are such that \(a + bq\) is right-invertible for some \(q,\) then \((a, b)\) is a Euclidean pair. Hence if \(R\) is of stable range one, then every pair \((a, b)\) with \(aR + bR = R\) is Euclidean.

4. If \(e = e^2\) is such that \(eRe = Re\) (\(e\) is said to be left semi central) then for any \(b \in R, (e, b)\) is a Euclidean pair.
**Definition 1.3.** A ring $R$ is a right $K$-Hermite ring if for any $(a, b) \in R^2$ there exists an invertible $2 \times 2$ matrix $P \in GL_2(R)$ and an element $d \in R$ such that $(a, b)P = (d, 0)$.

**Theorem 1.4.** Let $a, b$ be elements in a ring $R$. The following are equivalent:

1. $(a, b)$ is a Euclidean pair.

2. For some $n \geq 0$ there exist $q_1, \ldots, q_{n+1} \in R$ and $r_n \in R$ such that

   
   $$(a, b) = (r_n, 0)P(q_{n+1}) \cdots P(q_1).$$

   In particular, every right quasi-Euclidean ring is right $K$-Hermite.

3. For some $n \geq 0$ there exist $q_1, \ldots, q_{n+1} \in R$ and $r_n \in R$ such that $a = r_n p_{n+1} (q_{n+1}, \ldots, q_1)$ and $b = r_n p_n (q_{n+1}, \ldots, q_2)$.

   Now, let $(a, b) \in R^2$ be a Euclidean pair. Then

   a) $aR + bR = r_n R$ where $r_n$ is the last nonzero remainder of the Euclidean algorithm.

   b) If $r_n$ is either central or not a left zero-divisor in $R$, then $aR \cap bR$ is also principal.

   c) $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of $n + 2$ idempotents in $M_2(R)$.

**Proof.** Sketch of partial proof of (c) above (n=1):

Want to show that if

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} P(q_2)P(q_1)$$

then the matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of idempotents.

Write successively

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q_2 & 1 \end{pmatrix} P(q_1)$$

Notice that the second matrix of the RHS is an idempotent. Conjugating with the last matrix $P(q_1)$ we get

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} P(q_1)P(q_1)^{-1} \begin{pmatrix} 0 & 0 \\ q_2 & 1 \end{pmatrix} P(q_1)$$
and so,
\[
\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} P(q_1)^{-1} \begin{pmatrix} 0 & 0 \\ q_2 & 1 \end{pmatrix} P(q_1)
\]

More generally
\[
a = bq_1 + r_1, \ b = r_1q_2 + r_2, \ r_1 = r_2q_3 + r_3, \ldots, r_{n-1} = r_nq_{n+1}.
\]

Let us define:
\[
Q_i = \begin{pmatrix} q_i & 1 \\ 1 & 0 \end{pmatrix}, \ E_i = \begin{pmatrix} 0 & 0 \\ q_i + 1 & 1 \end{pmatrix}, \ P_i = Q_iQ_{i-1} \cdots Q_1
\]

We then have:
\[
\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} E_1 E_2 P_1 E_3 P_2 \cdots E_{n+1} P_n
\]

\[\square\]

**Examples 1.5.** (1) Let \((a, b) = (14, 8)\) over \(R = \mathbb{Z}\), for which \(n = 2\), \(q_1 = q_2 = 1\), \(q_3 = 3\), and \(r_2 = \gcd(14, 8) = 2\). Applying (c) above we get the following factorization of \(A\) into \(n + 2 = 4\) idempotents:

\[
A = \begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}).
\]

Not unique: here is a shorter factorization:

\[
\begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}),
\]

and it can be shown that this is in fact “a shortest” factorization for \(A\).

(2) Statement (c) is only a necessary but not a sufficient condition for \((a, b)\) to be a Euclidean pair. To see this, let \(\theta = \sqrt{-5}\) and \(R = \mathbb{Z}[\theta]\). The ideal \(-2R + (\theta + 1) R\) is not principal.

The matrix \(E = \begin{pmatrix} -2 & \theta + 1 \\ \theta - 1 & 3 \end{pmatrix}\) over \(R\) has trace 1 and determinant 0, so \(E^2 = E\).
Thus, \( A := \begin{pmatrix} -2 & \theta + 1 \\ 0 & 0 \end{pmatrix} = \text{diag}(1,0) E \). However, the ideal \(-2R + (\theta + 1) R\) is not a principal ideal. In particular, \((-2, \theta + 1)\) is not a Euclidean pair over \( R \), according to Theorem 1.4 (3),(a).

(3) If the pair \((a, b)\) is left Euclidean instead, a similar decomposition into products of idempotents holds for the matrix \( \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \).

## 2 Euclidean pairs and Euclidean rings

**Definitions 2.1.**
1. \( R \) is of stable range one if \( aR + bR = R \) implies that there exists \( x \in R \) such that \( a + bx \) is invertible in \( R \).
2. \( R \) is a right Bézout ring if finitely generated ideals are principal.
3. \( R \) is projective free if projective finitely generated right \( R \)-modules are free.
4. \( R \) is a \( GE_2 \)-ring if \( GL_2(R) \) is generated by elementary matrices and invertible diagonal matrices.

**Theorem 2.2.** Let \( R \) be a ring of stable range 1. Then \((a, b) \in R^2\) is a Euclidean pair if and only if the right ideal \( aR + bR \) is principal.

In particular:
1. If \( R \) is a right Bézout ring with stable range 1 (e.g. \( R \) can be any semilocal right Bézout ring), then \( R \) is right quasi-Euclidean.
2. If \( R \) is a unit-regular ring, then all matrix rings \( M_n(R) \) are right (and left) quasi-Euclidean.

**Proof.** Proof of the first statement:

The “only if” part is Theorem above (a).

For the “if” part, assume that \( aR + bR = dR \) for some \( d \in R \), and write \( a = da_0, \ b = db_0, \) and \( d = ax + by \). Letting \( c = 1 - a_0x - b_0y \), we have \( dc = d - ax - by = 0 \), and \( a_0x + (b_0y + c) = 1 \). Since \( R \) has stable range 1, there exists \( t \in R \) such that \( u := a_0 + (b_0y + c) t \) is a unit. Left-multiplication by \( d \) then yields \( du = a + byt + dct = a + byt \).

We have now \( a = b(-yt) + du \) and \( b = (du)(u^{-1}b_0) \), so \((a, b)\) is a Euclidean pair. \( \Box \)
Theorem 2.3. For any ring $R$, the following statements are equivalent:

(A) $R$ is right quasi-Euclidean.
(B) $R$ is a GE-ring that is right K-Hermite.
(C) $R$ is a GE$_2$-ring that is right K-Hermite.
(D) For any $a, b \in R$, $(a, b) = (r, 0)Q$ for some $r \in R$ and $Q \in GE_2(R)$.
(E) For any $a, b \in R$, $(a, b) = (r, 0)Q$ for some $r \in R$ and $Q \in E_2(R)$.

If $R$ is a domain there is another characterization. Recall that $R$ is a projective-free if every finitely generated projective module is free.

Theorem 2.4. A domain $R$ is right quasi-Euclidean if and only if $R$ is a projective-free GE$_2$-ring such that every matrix \( \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \) is a product of idempotents in $M_2(R)$.

As an application of Theorem 2.3 we obtain the following results:

Theorem 2.5. 1. If $R$ is a right quasi-Euclidean ring, so is $S = M_k(R)$ for every $k \geq 1$.

2. For any ideal $I \subseteq \text{rad } (R)$, $R$ is right quasi-Euclidean if and only if $R$ is right Bézout and $R/I$ is right quasi-Euclidean.

3. If $R$ is a right Euclidean ring and $S$ is a right denominator set then $RS^{-1}$ is right Euclidean.

3 Left-Right Symmetry and Dedekind-Finiteness

Example 3.1. $k$ a field and $a \in k \setminus \sigma(k)$ a non-surjective endomorphism of $k$. $R$ stands for $R = k[x; \sigma]$.

* $R = k[x; \sigma]$ is a left Euclidean domain with respect to the usual degree function; in particular, $R$ is a left quasi-Euclidean domain.

* One can check that $axR \cap xR = 0$, and that the right ideal direct sum $axR + xR$ is non-principal.
* $R$ is not right Bézout hence not a right quasi-Euclidean domain.
* $(ax, x)$ is a left Euclidean pair but it is not a right Euclidean pair.
* $R$ is a left PID hence it is a projective-free ring; Thus, by a previous lemma, the fact that $axR + xR$ is non-principal implies that the matrix $A = \begin{pmatrix} ax & x \\ 0 & 0 \end{pmatrix}$ is not a product of idempotent matrices over $R$.
* for any two elements $a, x$ in any ring, the “other” pair $(xa, x)$ is obviously always a right Euclidean pair and indeed the matrix $B = \begin{pmatrix} xa & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$ is a product of idempotent matrices.

but under some circumstances there is a symmetry:

**Theorem 3.2.** (A) A left quasi-Euclidean ring $R$ is right quasi-Euclidean if and only if it is right Bézout. In particular, a regular ring is left quasi-Euclidean if and only if it is right quasi-Euclidean.

(B) A left quasi-Euclidean domain $R$ is right quasi-Euclidean if and only if it is a right Ore domain.

A right Euclidean is not necessarily Dedekind finite ($ab = 1 \Rightarrow ba \neq 1$).

**Example 3.3.** (due to Bergman)

Let $A = k[[x]]$ over a field $k$, and let $K = k((x))$ be the Laurent series field, which is the quotient field of $A$.

(A) $R = \{ f \in \text{End}_k(A) : \exists f_0 \in K$ such that $(f - f_0)(x^nA) = 0$ for some $n \geq 1 \}$. $R$ is von Neuman regular but not Dedekind finite.

Steps to prove that $R$ is right Euclidean (and hence left Euclidean):

(B) For any $f, g \in R$,

\[ f \in Rg \iff \ker(g) \subseteq \ker(f) \]

(C) If $n$ is chosen large enough so that $x^nA \cap \text{im}(g) = 0$. Then

\[ \ker(g + x^n f) = \ker(f) \cap \ker(g) \]
4 Applications.

A) Decomposition of singular matrices

Theorem 4.1. Let $R$ be a right quasi-Euclidean domain and let $A \in M_2(R)$ be such that $\text{l.ann}(A) \neq 0$. Then $A$ is a product of idempotent matrices.

Proposition 4.2. Let $R$ be a right quasi-Euclidean domain and $A \in M_n(R)$. Then $\text{l.ann}(A) \neq 0$ implies that $\text{r.ann}(A) \neq 0$.

Theorem 4.3. Let $R$ be a right and left quasi-Euclidean domain. Then every matrix $A \in M_n(R)$ with $\text{l.ann}(A) \neq 0$ (equivalently, $\text{r.ann}(A) \neq 0$) is a product of idempotent matrices.

A ring has the IP property if any singular matrix is a product of idempotent matrices. A ring has the IP$_2$ property if every $2 \times 2$ singular matrix is a product of idempotent matrices.

Corollaire 4.4. Let $R$ be a domain which is any one of the following types:

(a) a Euclidean domain,

(b) a local domain such that its radical $J = Rg = gR$ with $\cap Rg^n = 0$,

(c) a commutative principal ideal domain with IP$_2$, or

(d) a local Bézout domain.

Then every singular matrix over $R$ is a product of idempotent matrices (in other words, $R$ has the IP property).
B) Rings with the SSP property.

More Euclidean pairs:

Using the fact that for a Euclidean pair \((a, b)\), \(aR + bR\) is principal, one can show the following Theorem.

**Theorem 4.5.** For a ring \(R\) the following are equivalent

(i) \(\text{idem}(R).\text{idem}(R) \subseteq \text{reg}(R)\).

(ii) \(\text{reg}(R)\).\text{reg}(R) \subseteq \text{reg}(R)\).

(iii) \(\text{ureg}(R).\text{ureg}(R) \subseteq \text{reg}(R)\).

(iv) \(R_R\) satisfies the SSP property.

(v) \(R_R\) satisfies the SSP property.

In a ring \(R\) which satisfies one of these equivalent statement one can show that a pair \((a, b)\) where \(a \in \text{ureg}(R)\) and \(b \in \text{reg}(R)\) is automatically a Euclidean pair. Thus if \(e\) is an idempotent in a regular ring then \((e, b)\) is an Euclidean pair for any \(b \in R\).

5 Continuant polynomials.

Recall \(p_n(t_1, t_2, \ldots, t_n) \in \mathbb{Z}\langle t_1, \ldots, t_n \rangle\) are such that \(p_0 = 1, p_1(t_1) = t_1, p_2(t_1, t_2) = t_1t_2 + 1, p_n(t_1, t_2, t_3) = t_1t_2t_3 + t_1 + t_3, \ldots\)

for \(n \geq 2, p_n(t_1, \ldots, t_n) = p_{n-1}(t_1, \ldots, t_{n-1})t_n + p_{n-2}(t_1, \ldots, t_{n-2})\)

They appear, for instance, in:

- Continued fractions
- Getting Generators for \(GL_2(R)\) (P.M. Cohn).
- Characterizations of comaximal relations in certain rings (P.M. Cohn).
- Characterizations of Euclidean pairs and quasi Euclidean rings.

We collect a bunch of relations for these polynomials

**Proposition 5.1.**

- \(p_n(t_1, \ldots, t_n) = t_1p_{n-1}(t_2, \ldots, t_n) + p_{n-2}(t_3, \ldots, t_n)\).
- \(p_n(0, t_2, \ldots, t_n) = p_{n-2}(t_3, \ldots, t_n)\).
\[ p_n(1, t_2, \ldots, t_n) = p_n(t_2 + 1, t_3, \ldots, t_n). \]

- for \( 1 \leq k \leq n \), we have
  \[ p_n(t_1, \ldots, t_n) = p_k(t_1, \ldots, t_k)p_{n-k}(t_{k+1}, \ldots, t_n) + p_{k-1}(t_1, \ldots, t_{k-1})p_{n-k-1}(t_{k+2}, \ldots, t_n). \]

- Relations coming from the fact that the inverse of \( P(t_1) \cdots P(t_n) \) is equal to \( P(0)P(-t_n) \cdots P(-t_1)P(0) \).

- For \( 1 \leq m \leq n \), one has
  \[ \frac{\partial p_n(t_1, \ldots, t_n)}{\partial t_m} = p_{m-1}(t_1, \ldots, t_{m-1})p_{n-m}(t_{m+1}, \ldots, t_n). \]

First leapfrog construction

0) The first term of \( p_n \) is \( t_1t_2 \cdots t_n \).

1) The next terms are obtained by erasing two consecutive indeterminates (the frog jumps over them) from \( t_1t_2 \cdots t_n \) to get the sum:
\[ t_3t_4 \cdots t_n + t_1t_4t_5 \cdots t_n + t_1t_2t_5 \cdots t_n + \ldots \]

2) We erase 2 pairs of consecutive indeterminates (2 jumps) and get the terms
\[ \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} t_1 \cdots \hat{t}_{i_1} \hat{t}_{i_1+1} \cdots \hat{t}_{i_2} \hat{t}_{i_2+1} \cdots t_n. \]

3) We then continue adding terms corresponding to 3 leaps, 4 leaps, and so on. Finally, we can write
\[ p_n(t_1, \ldots, t_n) = \sum_{i_1, i_2, \ldots, i_j} t_1 \cdots \hat{t}_{i_1} \hat{t}_{i_1+1} \cdots \hat{t}_{i_2} \hat{t}_{i_2+1} \cdots \hat{t}_{i_j} \hat{t}_{i_j+1} \cdots t_n \]
where \( 1 \leq j \leq \lfloor n/2 \rfloor \) and \( i_j + 1 < i_{j+1} \) for every \( j \),

Second leapfrog construction

Remark that
- \( p_{2n} \) is a sum of monomials with an even number of factors.
- \( p_{2n+1} \) is a sum of monomials with an odd number of factors.

Put \( x_n = t_{2n-1}, y_n = t_{2n} \) and \( G_n = p_{2n}, H_n = p_{2n-1} \).
So \( G_n \) is a polynomial in the indeterminates \( x_1, y_1, \ldots, x_n, y_n \), and \( H_n \) is a polynomial in the indeterminates \( x_1, y_1, \ldots, y_{n-1}, x_n \).
We have:

\[ G_0 = 1, \quad G_1 = x_1y_1 + 1, \quad G_2 = x_1y_1x_2y_2 + x_1y_1 + x_1y_2 + x_2y_2 + 1, \]
\[ G_3 = x_1y_1x_2y_2x_3y_3 + x_1y_1x_2y_2 + x_1y_1x_2y_3 + x_1y_1x_3y_3 + 
   x_1y_2x_3y_3 + x_2y_2x_3y_3 + x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 + 1 \]

and
\[ H_0 = 0, \quad H_1 = x_1, \quad H_2 = x_1y_1x_2 + x_1 + x_2, \]
\[ H_3 = x_1y_1x_2y_2x_3 + x_1y_1x_2 + x_1y_1x_3 + x_1y_2x_3 + x_2y_2x_3 + x_1 + x_2 + x_3. \]

Now consider the following directed graph (quiver) \( \Gamma_n \) with two vertices \( A \) and \( B \):

```
A ← x_2 → B ← y_2 → A ← x_1 → B ← y_1 → A
```

Thus \( \Gamma_n \) has \( 2n \) arrows, of which \( n \) goes from \( A \) to \( B \) and are indexed by the indeterminates \( x_i \), and \( n \) from \( B \) to \( A \) indexed by the indeterminates \( y_i \).

Let \( k \) be a field, consider the quiver algebra \( k\Gamma_n \) and the ideal \( I \) of \( k\Gamma_n \) generated by all paths \( x_iy_j : A \xrightarrow{x_i} B \xrightarrow{y_j} A \) with \( i > j \) and all paths \( y_ix_j : B \xrightarrow{y_i} A \xrightarrow{x_j} B \) with \( i \geq j \).

**Theorem 5.2.** Let \( R = k\Gamma_n/I \).

1) The \( k \)-algebra \( R \) is finite dimensional.

2) The Jacobson radical \( J(R) \) is a nilpotent ideal that contains all nilpotent elements of \( R \).

3) \( R = R_0 \oplus R_1 \) is 2-graded, where \( R_0 \) corresponds to the paths of even length and \( R_1 \) to the paths of odd length.

4) The images of the polynomials \( G_n \) in \( R \) are in \( R_0 \) and the images of the polynomials \( H_n \) are in \( R_1 \).

5) \n\[ H_n = \left( 1 - \sum_{1 \leq i \leq j \leq n} x_iy_j \right)^{-1} \left( \sum_{i=1} x_i \right) \quad \text{and} \quad G_n = \left( 1 - \sum_{1 \leq i \leq j \leq n} x_iy_j \right)^{-1} \]
for every \( n \geq 0 \).
6 Generalized Fibonacci Polynomials

Definition 6.1. The polynomials \( f_n \in \mathbb{Z}\langle x_1, y_1, x_2, y_2, \ldots \rangle \) are defined by the recursion formulae:

\[
\begin{align*}
&f_{-1} = 0, \quad f_0 = 1, \\
&f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})x_n + \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + f_{n-2}(x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2})y_n.
\end{align*}
\]

The first of these polynomials \( f_n \) are

\[
\begin{align*}
f_0 &= 1, \quad f_1 = x_1, \quad f_2 = x_1x_2 + y_2, \\
f_3 &= x_1x_2x_3 + x_1y_3 + y_2x_3, \\
f_4 &= x_1x_2x_3x_4 + x_1x_2y_4 + x_1y_3x_4 + y_2x_3x_4 + y_2y_4, \\
f_5 &= x_1x_2x_3x_4x_5 + x_1x_2x_3y_5 + x_1x_2y_4x_5 + x_1y_3x_4x_5 + \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + x_1y_3y_5 + y_2x_3x_4x_5 + y_2x_3y_5 + y_2y_4x_5, \ldots
\end{align*}
\]

- The number of monomials in each \( f_n \) is the \((n+1)\)th Fibonacci number \( F_{n+1} \).
- When we specialize all the indeterminates \( y_i \) to 1, we get back the continuant polynomials i.e. \( f_n(x_1, \ldots, x_n, 1, \ldots, 1) = p_n(x, \ldots, x_n) \).
- If we specialize further: \( f_n(x, \ldots, x, 1, 1, \ldots, 1) = F_n(x) \), i.e. we get the commutative Fibonacci polynomials.
- The polynomials \( f_n \) are homogeneous of degree \( n \) if we give the \( x_i \) degree one and the \( y_i \) degree 2.
- Notice that the indeterminate \( y_1 \) does not appear in any polynomial \( f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) \).

Theorem 6.2. 1. \( f_n(2, 2, \ldots, 2, -1, -1, \ldots, -1) = n \)

2. \( f_n(x+1, x+1, \ldots, x+1, -x, -x, \ldots, -x) = 1 + x + x^2 + \cdots + x^{n-1} \).

3. We have:

\[
\mathcal{F}_n := \begin{pmatrix} x_1 & 1 \\ y_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} = \\
\begin{pmatrix} f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) & f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) \\ y_1f_{n-1}(x_2, \ldots, x_n, y_2, \ldots, y_n) & y_1f_{n-2}(x_2, \ldots, x_{n-1}, y_2, \ldots, y_{n-1}) \end{pmatrix}.
\]
4. 
\[ \mathcal{F}_n = \begin{pmatrix}
    f_k(x_1, \ldots, y_k) & f_{k-1}(x_1, \ldots, y_{k-1}) \\
    y_1 f_{k-1}(x_2, \ldots, y_k) & y_1 f_{k-2}(x_2, \ldots, y_{k-1}) \\
    f_{n-k}(x_{k+1}, \ldots, y_n) & f_{n-k-1}(x_{k+1}, \ldots, y_{n-1}) \\
    y_{k+1} f_{n-k-1}(x_{k+2}, \ldots, y_n) & y_{k+1} f_{n-k-2}(x_{k+2}, \ldots, y_{n-1})
\end{pmatrix}. \]

5. \( f_n(x_1, \ldots, x_n, y_1, x_1 x_2, x_2 x_3, x_3 x_4, \ldots, x_{n-1} x_n) = F_{n+1} x_1 x_2 \ldots x_n. \)

6. \( f_n(x_1, \ldots, y_n) = f_k(x_1, \ldots, y_k) f_{n-k}(x_{k+1}, \ldots, y_n) + \\
   + f_{k-1}(x_1, \ldots, y_{k-1}) y_{k+1} f_{n-k-1}(x_{k+2}, \ldots, y_n) \)

7. \( f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = \\
   = x_1 f_{n-1}(x_2, \ldots, x_n, y_2, \ldots, y_n) + y_2 f_{n-2}(x_3, \ldots, x_n, y_3, \ldots, y_n). \)

8. \( f_n(x_1, x_2, \ldots, y_n) = \\
   = f_{k+1}(x_1, \ldots, x_k, f_{n-k}(x_{k+1}, \ldots, y_n), y_1, \ldots, y_k, f_{n-k-1}(x_{k+2}, \ldots, y_n)). \)

9. \( \frac{\partial f_n(x_1, \ldots, y_n)}{\partial x_k} = f_{k-1}(x_1, \ldots, y_{k-1}) f_{n-k}(x_{k+1}, \ldots, y_n), \text{ for } 1 \leq k \leq n. \)

\( \frac{\partial f_n(x_1, \ldots, y_n)}{\partial y_k} = f_{k-2}(x_1, \ldots, y_{k-2}) f_{n-k}(x_{k+1}, \ldots, y_n), \text{ for } 2 \leq k \leq n. \)

It is also possible to describe the generalized Fibonacci polynomials via leapfrog constructions and a path algebra can also be defined based on this definition.

7  Tilings and general recurrence sequences.

**Definition 7.1.** A linear tiling of a row of squares (a \( 1 \times n \) strip of square cells) is a covering of the strip of squares with squares and dominos (which cover two squares).

For instance, the polynomial \( f_3 = x_1 x_2 x_3 + x_1 y_3 + y_2 x_3 \) parametrizes the set of the three linear tilings

![Tilings](image)

of a row of three squares. Here \( x_i \) denotes the \( i \)-th square and \( y_i \) denotes the domino that covers the \( (i-1) \)-th and the \( i \)-th square (the
The Fibonacci number $F_n$ represents the number of tilings of a strip of length $n$ using length 1 squares and length 2 dominos.

Now consider the following family of polynomials $g_n$, with $n \geq 0$. To define them, we need countably many non-commutative indeterminates $x_{ij}$, where $1 \leq i \leq j$. Set $g_0 = 1$ and

$$g_n = \sum_{i=1}^{n} g_{i-1} x_{in}, \text{ for } n \geq 1.$$  

For instance, the first polynomials $g_n$ are

$g_1 = x_{11}$,  
$g_2 = x_{12} + x_{11} x_{22}$,  
$g_3 = x_{13} + x_{11} x_{23} + x_{12} x_{33} + x_{11} x_{22} x_{33}$,  
$g_4 = x_{14} + x_{11} x_{24} + x_{12} x_{34} + x_{11} x_{22} x_{34} + x_{13} x_{44} + x_{11} x_{23} x_{44} + x_{12} x_{33} x_{44} + x_{11} x_{22} x_{33} x_{44}$.

For every $n \geq 1$, the polynomial $g_n \in \mathbb{Z}\langle x_{ij} \mid 1 \leq i \leq j \leq n \rangle$. The polynomial $g_n$ is a sum of monic monomials that parametrize all linear tilings of a strip of $n$ square cells, that is, all coverings of the strip of squares with rectangles of any length 1, 2, ..., $n$. The indeterminate $x_{ij}$ indicates the rectangle of length $j - i + 1$ that starts from the $i$-th square and ends covering the $j$-th square.

For instance, $g_3 = x_{13} + x_{11} x_{23} + x_{12} x_{33} + x_{11} x_{22} x_{33}$ and, correspondingly, the tilings of a strip of three squares are

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We can get back the polynomials $p_n$ and $f_n$ by different specializations.

We have:

$$(g_1, \ldots, g_n) = (g_0, \ldots, g_{n-1}) \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ 0 & x_{22} & \ldots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & x_{nn} \end{pmatrix}$$

Since, for $1 \leq l \leq n$, a tiling of a strip of length $n$ is obtained by a tile of length $l$ followed by a tiling of length $n - l$, the following formula,
where we have specified explicitly the indeterminates ("the tiles") for each polynomial, is easy to get:

\[ g_n(x_{ij}; 1 \leq i \leq j \leq n) = \sum_{l=1}^{n} x_{1l} g_{n-l}(x_{l+i,l+j}; 1 \leq i \leq j \leq n-l) \]

Let \( R \) be a ring, define a mapping \( \text{perm}: M_n(R) \to R \) setting, for every matrix \( A = (a_{i,j})_{i,j} \in M_n(R) \),

\[ \text{perm}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}. \]

If \( A_{i,j} \) denotes the \((n-1) \times (n-1)\)-matrix that results from \( A \) removing the \( i \)-th row and the \( j \)-th column, then \( \text{perm}(A) := \sum_{j=1}^{n} a_{1,j} \text{perm}(A_{1,j}) = \sum_{j=1}^{n} \text{perm}(A_{n,j}) a_{n,j} \) (it is possible to easily expand our permanent along the first row or the last row only).

**Theorem 7.2.** For every \( n \geq 1 \), we have:

\[ g_n(x_{ij}) = \text{perm}(A_n) = \text{perm}(A_n^t), \]

where

\[
A_n = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
1 & x_{22} & x_{23} & \cdots & x_{2n} \\
0 & 1 & x_{33} & \cdots & x_{3n} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & x_{nn}
\end{pmatrix}
\]

**References**


THANK YOU!