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Euclidean pairs, Quasi Euclidean rings and Continuant Polynomials

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Joint works with **A. Alahmadi**, **S.K. Jain**, **T.Y.Lam** and **A. Facchini**

In all the talk R will stand for a unital associative ring.

1 The Euclidean pair (a,b) and its associated continuant polynomials.

Definitions 1.1. (a) An ordered pair $(a,b)\in R^2$ is a *right Euclidean pair* if there exist elements (q_1,r_1), ..., $(q_{n+1},r_{n+1})\in R^2$ (for some $n\geq 0$) such that $a=bq_1+r_1,\ b=r_1q_2+r_2$, and

$$(*)\qquad r_{i-1}=r_iq_{i+1}+r_{i+1}\ \text{for}\ 1<i\leq n,\ \text{with}\ r_{n+1}=0.$$

The notion of a *left Euclidean pair* is defined similarly.

(b) A ring R is right quasi Euclidean if every pair $(a,b)\in R^2$ is right Euclidean.

(c) Let $T=\{t_1,t_2,\dots\}$ be a countable set of noncommuting variables, and let $\mathbb{Z}\langle T\rangle$ be the free $\mathbb{Z}$-algebra generated by T. We define the n-th *right continuant polynomials*

$$p_n(t_1,\dots,t_n)\in\mathbb{Z}\langle t_1,\dots,t_n\rangle\subseteq\mathbb{Z}\langle T\rangle$$

by $p_0=1,\ p_1(t_1)=t_1$, and inductively for $i\geq 2$ by

$$p_i(t_1,\dots,t_i)=p_{i-1}(t_1,\dots,t_{i-1})\,t_i+p_{i-2}(t_1,\dots,t_{i-2}).$$

Thus, $p_2(t_1,t_2)=t_1t_2+1,\ p_3(t_1,t_2,t_3)=t_1t_2t_3+t_3+t_1$, etc.

Notation: $P(q)=\begin{pmatrix}q&1\\1&0\end{pmatrix}$

Are there connections between these three notions ?

Let us consider an easy example:

$(a,b)=(22,8)\in\mathbb{Z}^2$ we write

$22=8\times 2+6$

$8=6\times 1+2$

$6=2\times 3$

we then have:

$$(22,8)=(8,6)\begin{pmatrix}2&1\\1&0\end{pmatrix}$$

$$(22,8) = (6,2)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}$$

$$(22,8) = (2,0)\begin{pmatrix}3 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}$$

In general for a Euclidean pair (a,b) $a = bq_1 + r_1,\ b = r_1q_2 + r_2$, and

$$(*) \qquad r_{i-1} = r_iq_{i+1} + r_{i+1} \text{ for } 1 < i \leq n, \text{ with } r_{n+1} = 0.$$

We will get that

$$(a,b) = (r_n, 0)P(q_{n+1})P(q_n)\cdots P(q_1)$$

Now, looking at a the product $P(t_1)P(t_2)\cdots P(t_n)$
we have

$$P(t_1)P(t_2) = \begin{pmatrix}t_1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t_2 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}t_1t_2+1 & t_1\\ t_2 & 1\end{pmatrix}$$

and

$$P(t_1)P(t_2)P(t_3) = \begin{pmatrix}t_1t_2+1 & t_1\\ t_2 & 1\end{pmatrix}\begin{pmatrix}t_3 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}t_1t_2t_3+t_1+t_3 & t_1t_2+1\\ t_2t_3+1 & t_2\end{pmatrix}$$

In general:

$$P(t_1)P(t_2)\cdots P(t_n) = \begin{pmatrix}p_n(t_1,\dots,t_n) & p_{n-1}(t_1,\dots,t_{n-1})\\ p_{n-2}(t_2,\dots,t_n) & p_{n-2}(t_2,\dots,t_{n-1)}\end{pmatrix}$$

Examples 1.2. 1. $(bq,b), (a,0)$ are Euclidean pairs for any $a,b,q \in R$.

2. If (a,b) is a Euclidean pair and $c \in R$ then (b,a), (ca,cb), $(ac+b,a)$, $(bc+a,b)$ are Euclidean pairs.

3. If $a,b \in R$ are such that $a+bq$ is right-invertible for some q, then (a,b) is a Euclidean pair. Hence if R is of stable range one, then every pair (a,b) with $aR+bR=R$ is Euclidean.

4. If $e = e^2$ is such that $eRe = Re$ (e is said to be left semi central) then for any $b \in R$, (e,b) is a Euclidean pair.

Definition 1.3. A ring R is a right K-Hermite ring if for any $(a,b)\in R^2$ there exists an invertible 2×2 matrix $P\in GL_2(R)$ and an element $d\in R$ such that $(a,b)P=(d,0)$.

Theorem 1.4. *Let a,b be elements in a ring R. The following are equivalent:*

(1) (a,b) is a Euclidean pair.

(2) For some $n\geq 0$ there exist $q_1,\dots,q_{n+1}\in R$ and $r_n\in R$ such that

$$(a,b)=(r_n,0)\,P(q_{n+1})\cdots P(q_1).$$

In particular, every right quasi-Euclidean ring is right K-Hermite.

(3) For some $n\geq 0$ there exist $q_1,\dots,q_{n+1}\in R$ and $r_n\in R$ such that $a=r_np_{n+1}\,(q_{n+1},\dots,q_1)$ and $b=r_np_n\,(q_{n+1},\dots,q_2)$.

Now, let $(a,b)\in R^2$ be a Euclidean pair. Then

(a) $aR+bR=r_nR$ where r_n is the last nonzero remainder of the Euclidean algorithm.

(b) If r_n is either central or not a left zero-divisor in R, then $aR\cap bR$ is also principal.

(c) $\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}$ is a product of $n+2$ idempotents in $\mathbb{M}_2(R)$.

Proof. Sketch of partial proof of (c) above (n=1):

Want to show that if

$$\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix} r & 0\\ 0 & 0\end{pmatrix}P(q_2)P(q_1)$$

then the matrix $\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}$ is a product of idempotents.

Write successively

$$\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 1 & r\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ q_2 & 1\end{pmatrix}P(q_1)$$

Notice that the second matrix of the RHS is an idempotent. Conjugating with the last matrix $P(q_1)$ we get

$$\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 1 & r\\ 0 & 0\end{pmatrix}P(q_1)P(q_1)^{-1}\begin{pmatrix} 0 & 0\\ q_2 & 1\end{pmatrix}P(q_1)$$

and so,

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ q_1 + r & 1 \end{pmatrix} P(q_1)^{-1} \begin{pmatrix} 0 & 0 \\ q_2 & 1 \end{pmatrix} P(q_1)$$

More generally

$$a = bq_1 + r_1,\ b = r_1 q_2 + r_2,\ r_1 = r_2 q_3 + r_3, \ldots, r_{n-1} = r_n q_{n+1}.$$

Let us define:

$$Q_i = \begin{pmatrix} q_i & 1 \\ 1 & 0 \end{pmatrix} \quad E_i = \begin{pmatrix} 0 & 0 \\ q_i + 1 & 1 \end{pmatrix} \quad P_i = Q_i Q_{i-1} \cdots Q_1$$

We then have:

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} E_1 E_2^{P_1} E_3^{P_2} \cdots E_{n+1}^{P_n}$$

□

Examples 1.5. (1) Let $(a, b) = (14, 8)$ over $R = \mathbb{Z}$, for which $n = 2$, $q_1 = q_2 = 1$, $q_3 = 3$, and $r_2 = \gcd(14, 8) = 2$. Applying (c) above we get the following factorization of A into $n + 2 = 4$ idempotents:

$$A = \begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}).$$

Not unique: here is a shorter factorization:

$$\begin{pmatrix} 14 & 8 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 14 & 8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}),$$

and it can be shown that this is in fact "a shortest" factorization for A.

(2) Statement (c) is only a necessary but *not a sufficient condition* for (a, b) to be a Euclidean pair. To see this, let $\theta = \sqrt{-5}$ and $R = \mathbb{Z}[\theta]$. The ideal $-2R + (\theta + 1) R$ is not principal.

The matrix $E = \begin{pmatrix} -2 & \theta + 1 \\ \theta - 1 & 3 \end{pmatrix}$ over R has trace 1 and determinant 0, so $E^2 = E$.

Thus, $A := \begin{pmatrix} -2 & \theta+1 \\ 0 & 0 \end{pmatrix} = \operatorname{diag}(1,0)\, E$. However, the ideal $-2R + (\theta+1)\, R$ is *not* a principal ideal. In particular, $(-2,\, \theta+1)$ is not a Euclidean pair over R, according to Theorem 1.4 (3),(a).

(3) If the pair (a,b) is *left* Euclidean instead, a similar decomposition into products of idempotents holds for the matrix $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$.

2 Euclidean pairs and Euclidean rings

Definitions 2.1. 1. R is of stable range one if $aR + bR = R$ implies that there exists $x \in R$ such that $a + bx$ is invertible in R.

2. R is a right Bézout ring if finitely generated ideals are principal.
3. R is projective free if projective finitely generated right R-modules are free.
4. R is a GE_2-ring if $GL_2(R)$ is generated by elementary matrices and invertible diagonal matrices.

Theorem 2.2. *Let R be a ring of stable range* 1*. Then $(a,b) \in R^2$ is a Euclidean pair if and only if the right ideal $aR + bR$ is principal.*

In particular:

(1) *If R is a right Bézout ring with stable range* 1 (*e.g.* R *can be any semilocal right Bézout ring*), *then R is right quasi-Euclidean.*

(2) *If R is a unit-regular ring, then all matrix rings $\mathbb{M}_n(R)$ are right* (*and left*) *quasi-Euclidean.*

Proof. Proof of the first statement:

The "only if" part is Theorem above (a).

For the "if" part, assume that $aR + bR = dR$ for some $d \in R$, and write $a = da_0$, $b = db_0$, and $d = ax + by$. Letting $c = 1 - a_0x - b_0y$, we have $dc = d - ax - by = 0$, and $a_0x + (b_0y + c) = 1$. Since R has stable range 1, there exists $t \in R$ such that $u := a_0 + (b_0y + c)\, t$ is a unit. Left-multiplication by d then yields $du = a + byt + dct = a + byt$. We have now $a = b\,(-yt) + du$ and $b = (du)\,(u^{-1}b_0)$, so (a,b) is a Euclidean pair. □

Theorem 2.3. *For any ring R, the following statements are equivalent*:

(A) *R is right quasi-Euclidean.*
(B) *R is a* GE*-ring that is right K-Hermite.*
(C) *R is a* GE_2*-ring that is right K-Hermite.*
(D) *For any $a, b \in R$, $(a, b) = (r, 0)\, Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$.*
(E) *For any $a, b \in R$, $(a, b) = (r, 0)\, Q$ for some $r \in R$ and $Q \in \mathrm{E}_2(R)$.*

If R is a domain there is another characterization. Recall that R is a projective-free if every finitely generated projective module is free.

Theorem 2.4. *A domain R is right quasi-Euclidean if and only if R is a projective-free* GE_2*-ring such that every matrix* $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *is a product of idempotents in* $\mathbb{M}_2(R)$.

As an application of Theorem 2.3 we obtain the following results:

Theorem 2.5.
1. *If R is a right quasi-Euclidean ring, so is $S = \mathbb{M}_k(R)$ for every $k \geq 1$.*
2. *For any ideal $I \subseteq \mathrm{rad}\,(R)$, R is right quasi-Euclidean if and only if R is right Bézout and R/I is right quasi-Euclidean.*
3. *If R is a right Euclidean ring and S is a right denominator set then RS^{-1} is right Euclidean.*

3 Left-Right Symmetry and Dedekind-Finiteness

Example 3.1. k a field and $a \in k \setminus \sigma(k)$ a non-surjective endomorphism of k. R stands for $R = k[x;\sigma]$.

* $R = k[x;\sigma]$ is a left Euclidean domain with respect to the usual degree function; in particular, R is a left quasi-Euclidean domain.
* One can check that $axR \cap xR = 0$, and that the right ideal direct sum $axR + xR$ is *non-principal.*

* R is not right Bézout hence not a right quasi-Euclidean domain.
* $(ax,\, x)$ is a left Euclidean pair but it is *not* a right Euclidean pair.
* R is a left PID hence it is a projective-free ring; Thus, by a previous lemma, the fact that $axR + xR$ is non-principal implies that the matrix $A = \begin{pmatrix} ax & x \\ 0 & 0 \end{pmatrix}$ is not a product of idempotent matrices over R.
* *for any two elements* a, x in any ring, the "other" pair $(xa,\, x)$ is obviously always a right Euclidean pair and indeed the matrix $B = \begin{pmatrix} xa & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$ is a product of idempotent matrices.

but under some circumstances there is a symmetry:

Theorem 3.2. (A) *A left quasi-Euclidean ring R is right quasi-Euclidean if and only if it is right Bézout. In particular, a regular ring is left quasi-Euclidean if and only if it is right quasi-Euclidean.*
(B) *A left quasi-Euclidean domain R is right quasi-Euclidean if and only if it is a right Ore domain.*

A right Euclidean is not necessarily Dedekind finite ($ab = 1 \Rightarrow ba \neq 1$).

Example 3.3. (due to **Bergman**)

Let $A = k[[x]]$ over a field k, and let $K = k\,((x))$ be the Laurent series field, which is the quotient field of A.

(A) $R = \{\, f \in \mathrm{End}_k(A) : \exists\, f_0 \in K$ such that $(f - f_0)(x^n A) = 0$ for some $n \geq 1\}$. R is von Neuman regular but not Dedekind finite.

Steps to prove that R is right Euclidean (and hence left Euclidean):

(B) *For any* $f, g \in R$,

$$f \in R\,g \;\leftrightarrow\; \ker(g) \subseteq \ker(f)$$

.

(C) If n is chosen large enough so that $x^n A \cap \mathrm{im}\,(g) = 0$. Then

$$\ker(g + x^n f) = \ker(f) \cap \ker(g)$$

.

4 Applications.

A) **Decomposition of singular matrices**

Theorem 4.1. *Let R be a right quasi-Euclidean domain and let $A \in \mathbb{M}_2(R)$ be such that $l.ann\,(A) \neq 0$. Then A is a product of idempotent matrices.*

Proposition 4.2. *Let R be a right quasi-Euclidean domain and $A \in \mathbb{M}_n(R)$. Then $l.ann\,(A) \neq 0$ implies that $r.ann\,(A) \neq 0$.*

Theorem 4.3. *Let R be a right and left quasi-Euclidean domain. Then every matrix $A \in \mathbb{M}_n(R)$ with $l.ann\,(A) \neq 0$ (equivalently, $r.ann\,(A) \neq 0$) is a product of idempotent matrices.*

A ring has the IP property if any singular matrix is a product of idempotent matrices. A ring has the IP_2 property if every 2×2 singular matrix is a product of idempotent matrices.

Corollaire 4.4. *Let R be a domain which is any one of the following types:*

(a) a Euclidean domain,

(b) a local domain such that its radical $J = Rg = gR$ with $\cap Rg^n = 0$,

(c) a commutative principal ideal domain with IP_2, or

(d) a local Bézout domain.

Then every singular matrix over R is a product of idempotent matrices (in other words, R has the IP property).

B) **Rings with the SSP property**.

More Euclidean pairs:
Using the fact that for a Euclidean pair (a, b), $aR + bR$ is principal, one can show the following Theorem.

Theorem 4.5. *For a ring R the following are equivalent*

(i) $idem(R).idem(R) \subseteq reg(R)$.

(ii) $reg(R)reg(R) \subseteq reg(R)$.

(iii) $ureg(R).ureg(R) \subseteq reg(R)$.

(iv) R_R *satisfies the SSP property.*

(v) ${}_RR$ *satisfies the SSP property.*

In a ring R which satisfies one of these equivalent statement one can show that a pair (a, b) where $a \in ureg(R)$ and $b \in reg(R)$ is automatically a Euclidean pair. Thus if e is an idempotent in a regular ring then (e, b) is an Euclidean pair for any $b \in R$.

5 Continuant polynomials.

Recall $p_n(t_1, t_2, \ldots, t_n) \in \mathbb{Z}\langle t_1, \ldots, t_n\rangle$ are such that $p_0 = 1, p_1(t_1) = t_1, p_2(t_1, t_2) = t_1t_2 + 1,\ p_3(t_1, t_2, t_3) = t_1t_2t_3 + t_1 + t_3, \ldots$
for $n \geq 2$, $p_n(t_1, \ldots, t_n) = p_{n-1}(t_1, \ldots, t_{n-1})t_n + p_{n-2}(t_1, \ldots, t_{n-2})$
They appear, for instance, in:

- Continued fractions
- Getting Generators for $GL_2(R)$ (P.M. Cohn).
- Characterizations of comaximal relations in certain rings (P.M. Cohn).
- Characterizations of Euclidean pairs and quasi Euclidean rings.

We collect a bunch of relations for these polynomials

Proposition 5.1.

- $p_n(t_1, \ldots, t_n) = t_1p_{n-1}(t_2, \ldots, t_n) + p_{n-2}(t_3, \ldots, t_n)$.
- $p_n(0, t_2, \ldots, t_n) = p_{n-2}(t_3, \ldots, t_n)$.

- *$p_n(1, t_2, \dots, t_n) = p_{n_1}(t_2 + 1, t_3, \dots, t_n)$.*
- *for $1 \le k \le n$, we have $p_n(t_1, \dots, t_n) = p_k(t_1, \dots, t_k)p_{n-k}(t_{k+1}, \dots, t_n) + p_{k-1}(t_1, \dots, t_{k-1})p_{n-k-1}(t_{k+2}, \dots, t_n)$.*
- *Relations coming from the fact that the inverse of $P(t_1) \cdots P(t_n)$ is equal to $P(0)P(-t_n)P(-t_{n-1}) \cdots P(-t_1)P(0)$.*
- *For $1 \le m \le n$, one has $\frac{\partial p_n(t_1,\dots,t_n)}{\partial t_m} = p_{m-1}(t_1, \dots, t_{m-1})p_{n-m}(t_{m+1}, \dots, t_n)$.*

First leapfrog construction

0) The first term of p_n is $t_1t_2 \cdots t_n$.
1) The next terms are obtained by erasing two consecutive indeterminates (the frog jumps over them) from $t_1t_2 \cdots t_n$ to get the sum: $t_3t_4 \cdots t_n + t_1t_4t_5 \cdots t_n + t_1t_2t_5 \cdots t_n + \dots$.
2) We erase 2 pairs of consecutive indeterminates (2 jumps) and get the terms

$$\sum_{1 \le i_1 < i_2 - 1 \le n} t_1 \cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}} \cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}} \cdots t_n$$

.

3) We then continue adding terms corresponding to 3 leaps, 4 leaps, and so on. Finally, we can write

$$p_n(t_1, \dots, t_n) = \sum_{i_1, i_2, \dots, i_j} t_1 \cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}} \cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}} \cdots \widehat{t_{i_j}}\widehat{t_{i_j+1}} \cdots t_n$$

where $1 \le j \le \lfloor n/2 \rfloor$ and $i_j + 1 < i_{j+1}$ for every j,

Second leapfrog construction

Remark that

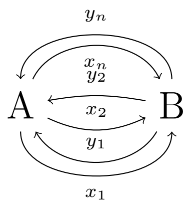
- p_{2n} is a sum of monomials with an even number of factors.
- p_{2n+1} is a sum of monomials with an odd number of factors.

Put $x_n = t_{2n-1},\ y_n = t_{2n}$ and $G_n = p_{2n},\ H_n = p_{2n-1}$.
So G_n is a polynomial in the indeterminates $x_1, y_1, \dots, x_n, y_n$, and H_n is a polynomial in the indeterminates $x_1, y_1, \dots, y_{n-1}, x_n$.

We have:

$G_0 = 1, \quad G_1 = x_1y_1 + 1, \quad G_2 = x_1y_1x_2y_2 + x_1y_1 + x_1y_2 + x_2y_2 + 1,$
$G_3 = x_1y_1x_2y_2x_3y_3 + x_1y_1x_2y_2 + x_1y_1x_2y_3 + x_1y_1x_3y_3+$
$\quad +x_1y_2x_3y_3 + x_2y_2x_3y_3 + x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 + 1$
and
$H_0 = 0, \quad H_1 = x_1, \quad H_2 = x_1y_1x_2 + x_1 + x_2,$
$H_3 = x_1y_1x_2y_2x_3 + x_1y_1x_2 + x_1y_1x_3 + x_1y_2x_3 + x_2y_2x_3 + x_1 + x_2 + x_3.$

Now consider the following directed graph (quiver) Γ_n with two vertices A and B:



Thus Γ_n has $2n$ arrows, of which n goes from A to B and are indexed by the indeterminates x_i, and n from B to A indexed by the indeterminates y_i.
Let k be a field, consider the quiver algebra $k\Gamma_n$ and the ideal I of $k\Gamma_n$ generated by all paths $x_iy_j \colon A \xrightarrow{x_i} B \xrightarrow{y_j} A$ with $i > j$ and all paths $y_ix_j \colon B \xrightarrow{y_i} A \xrightarrow{x_j} B$ with $i \geq j$.

Theorem 5.2. *Let $R = k\Gamma_n/I$.*

1) *The k-algebra R is finite dimensional.*
2) *The Jacobson radical $J(R)$ is a nilpotent ideal that contains all nilpotent elements of R.*
3) *$R = R_0 \oplus R_1$ is 2-graded, where R_0 corresponds to the paths of even length and R_1 to the paths of odd length.*
4) *The images of the polynomials G_n in R are in R_0 and the images of the polynomials H_n are in R_1.*
5)

$$H_n = \left(1 - \sum_{1\leq i\leq j\leq n} x_iy_j\right)^{-1} \left(\sum_{i=1} x_i\right) \quad \text{and} \quad G_n = \left(1 - \sum_{1\leq i\leq j\leq n} x_iy_j\right)^{-1}$$

for every $n \geq 0$.

6 Generalized Fibonacci Polynomials

Definition 6.1. The polynomials $f_n \in \mathbb{Z}\langle x_1, y_1, x_2, y_2, \ldots, \rangle$ are defined by the recursion formulae:
(6.I)

$$f_{-1} = 0, \quad f_0 = 1,$$
$$f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})x_n + \\ + f_{n-2}(x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2})y_n.$$

The first of these polynomials f_n are

$$f_0 = 1, \qquad f_1 = x_1, \qquad f_2 = x_1x_2 + y_2,$$
$$f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3,$$
$$f_4 = x_1x_2x_3x_4 + x_1x_2y_4 + x_1y_3x_4 + y_2x_3x_4 + y_2y_4,$$
$$f_5 = x_1x_2x_3x_4x_5 + x_1x_2x_3y_5 + x_1x_2y_4x_5 + x_1y_3x_4x_5 + \\ + x_1y_3y_5 + y_2x_3x_4x_5 + y_2x_3y_5 + y_2y_4x_5, \ldots$$

- The number of monomials in each f_n is the $(n+1)$-th Fibonacci number F_{n+1}.
- When we specialize all the indeterminates y_i to 1, we get back the continuant polynomials i.e. $f_n(x_1, \ldots, x_n, 1, \ldots, 1) = p_n(x_, \ldots, x_n)$.
- If we specialize further: $f_n(x, \ldots, x, 1, 1, \ldots, 1) = F_n(x)$, i.e. we get the commutative Fibonacci polynomials.
- The polynomials f_n are homogeneous of degree n if we give the x_i degree one and the y_i degree 2.
- Notice that the indeterminate y_1 does not appear in any polynomial $f_n(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Theorem 6.2. *1.* $f_n(2, 2, \ldots, 2, -1, -1, \ldots, -1) = n$

2. $f_n(x+1, x+1, \ldots, x+1, -x, -x, \ldots, -x) = 1 + x + x^2 + \cdots + x^{n-1}$.

3. We have:

$$\mathcal{F}_n := \begin{pmatrix} x_1 & 1 \\ y_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) & f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) \\ y_1 f_{n-1}(x_2, \ldots, x_n, y_2, \ldots, y_n) & y_1 f_{n-2}(x_2, \ldots, x_{n-1}, y_2, \ldots, y_{n-1}) \end{pmatrix}.$$

4.
$$\mathcal{F}_n = \begin{pmatrix} f_k(x_1,\ldots,y_k) & f_{k-1}(x_1,\ldots,y_{k-1}) \\ y_1 f_{k-1}(x_2,\ldots,y_k) & y_1 f_{k-2}(x_2,\ldots,y_{k-1}) \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} f_{n-k}(x_{k+1},\ldots,y_n) & f_{n-k-1}(x_{k+1},\ldots,y_{n-1}) \\ y_{k+1} f_{n-k-1}(x_{k+2},\ldots,y_n) & y_{k+1} f_{n-k-2}(x_{k+2},\ldots,y_{n-1}) \end{pmatrix}$$

5. $f_n(x_1,\ \ldots,\ x_n,\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4,\ \ldots,\ x_{n-1}x_n) = F_{n+1}x_1x_2\ldots x_n$.

6. $f_n(x_1,\ldots,y_n) = f_k(x_1,\ldots,y_k)f_{n-k}(x_{k+1},\ldots,y_n) +$
$+ f_{k-1}(x_1,\ldots,y_{k-1})y_{k+1}f_{n-k-1}(x_{k+2},\ldots,y_n)$

7. $f_n(x_1,\ldots,x_n,y_1,\ldots,y_n) =$
$= x_1f_{n-1}(x_2,\ldots,x_n,y_2,\ldots,y_n) + y_2f_{n-2}(x_3,\ldots,x_n,y_3,\ldots,y_n)$.

8. $f_n(x_1,x_2,\ldots,y_n) =$
$= f_{k+1}(x_1,\ldots,x_k,f_{n-k}(x_{k+1},\ldots,y_n),y_1,\ldots,y_k,f_{n-k-1}(x_{k+2},\ldots,y_n))$.

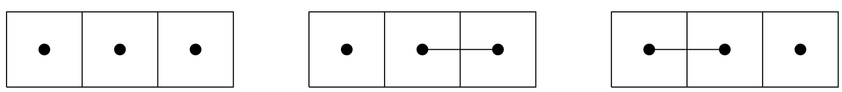
9. $\frac{\partial f_n(x_1,\ldots,y_n)}{\partial x_k} = f_{k-1}(x_1,\ldots,y_{k-1})f_{n-k}(x_{k+1},\ldots,y_n),\ \text{for } 1 \le k \le n.$
$\frac{\partial f_n(x_1,\ldots,y_n)}{\partial y_k} = f_{k-2}(x_1,\ldots,y_{k-2})f_{n-k}(x_{k+1},\ldots,y_n),\ \text{for } 2 \le k \le n.$

It is also possible to describe the generalized Fibonacci polynomials via leapfrog constructions and a path algebra can also be defined based on this definition.

7 Tilings and general recurrence sequences.

Definition 7.1. A *linear tiling* of a row of squares (a $1 \times n$ strip of square cells) is a covering of the strip of squares with squares and dominos (which cover two squares).

For instance, the polynomial $f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3$ parametrizes the set of the three linear tilings



of a row of three squares. Here x_i denotes the i-th square and y_i denotes the domino that covers the $(i-1)$-th and the i-th square (the domino that "ends on the i-th square".) The Fibonacci number F_n represents the number of tilings of a strip of length n using length 1 squares and length 2 dominos.

Now consider the following family of polynomials g_n, with $n \geq 0$. To define them, we need countably many non-commutative indeterminates x_{ij}, where $1 \leq i \leq j$. Set $g_0 = 1$ and

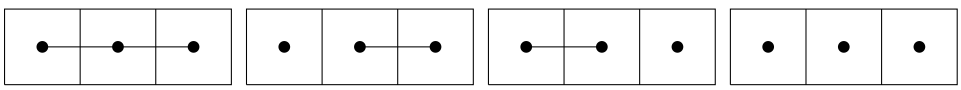
$$g_n = \sum_{i=1}^{n} g_{i-1} x_{in}, \quad \text{for} \quad n \geq 1. \tag{7.I}$$

For instance, the first polynomials g_n are

$$g_1 = x_{11}, \quad g_2 = x_{12} + x_{11}x_{22}, \quad g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33},$$
$$g_4 = x_{14} + x_{11}x_{24} + x_{12}x_{34} + x_{11}x_{22}x_{34} + x_{13}x_{44} + x_{11}x_{23}x_{44} + \\ + x_{12}x_{33}x_{44} + x_{11}x_{22}x_{33}x_{44}.$$

For every $n \geq 1$, the polynomial $g_n \in \mathbb{Z}\langle x_{ij} \,|\, 1 \leq i \leq j \leq n \rangle$. The polynomial g_n is a sum of monic monomials that parametrize all linear tilings of a strip of n square cells, that is, all coverings of the strip of squares with rectangles of any length $1, 2, \ldots, n$. The indeterminate x_{ij} indicates the rectangle of length $j - i + 1$ that starts from the i-th square and ends covering the j-th square.

For instance, $g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33}$ and, correspondingly, the tilings of a strip of three squares are



We can get back the polynomials p_n and f_n by different specializations.

We have:

$$(g_1, \ldots, g_n) = (g_0, \ldots, g_{n-1}) \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ 0 & x_{22} & \ldots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & x_{nn} \end{pmatrix}$$

Since, for $1 \leq l \leq n$, a tiling of a strip of length n is obtained by a tile of length l followed by a tiling of length $n - l$, the following formula, where we have specified explicitly the indeterminates (”the tiles”) for each polynomial, is easy to get:

$$g_n(x_{ij}; 1 \le i \le j \le n) = \sum_{l=1}^{n} x_{1l} g_{n-l}(x_{l+i,l+j}; 1 \le i \le j \le n-l)$$

R a ring, define a mapping perm: $M_n(R) \to R$ setting, for every matrix $A = (a_{i,j})_{i,j} \in M_n(R)$,

$$\operatorname{perm}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)}.$$

If $A_{i,j}$ denotes the $(n-1)\times(n-1)$-matrix that results from A removing the i-th row and the j-th column, then $\operatorname{perm}(A) := \sum_{j=1}^{n} a_{1,j}\operatorname{perm}(A_{1,j}) = \sum_{j=1}^{n} \operatorname{perm}(A_{n,j}) a_{n,j}$ (it is possible to easily expand our permanent along the first row or the last row only).

Theorem 7.2. *For every $n \ge 1$, we have:*

$$g_n(x_{ij}) = \operatorname{perm}(A_n) = \operatorname{perm}(A_n^t),$$

where

$$A_n = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 1 & x_{22} & x_{23} & \dots & x_{2n} \\ 0 & 1 & x_{33} & \ddots & x_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x_{nn} \end{pmatrix}$$

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THANK YOU !