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Continuant Polynomials

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Joint work with **Alberto Facchini**.

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Plan

1. Continuant polynomials: definition and interest.

2. Continuants and group of matrices.

3. Leapfrogs and graphs.

4. Generalized Fibonacci polynomials and tilings of strips.

5. General pattern and permanents.

1 Continuant polynomials: definition and interest.

$a, b \neq 0$ integers, suppose we have an Euclidean algorithm as follows:
$a = bq_1 + r_1; b = r_1q_2 + r_2; r_1 = r_2q_3 + r_3; r_2 = r_3q_4$ with $|r_i| > |r_{i+1}|$
We can then write:

$$\frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{b/r_1}$$

$$= q_1 + \cfrac{1}{q_2 + \cfrac{r_2}{r_1}} = q_1 + \cfrac{1}{q_2 + \frac{1}{r_1/r_2}}$$

$$= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \frac{r_3}{r_2}}}$$

$$= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \frac{1}{q_4}}}$$

going backwards we then write:

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \frac{q_4}{q_3q_4+1}} = q_1 + \frac{q_3q_4 + 1}{q_2q_3q_4 + q_2 + q_4}$$

and finally

$$\frac{a}{b} = \frac{q_1q_2q_3q_4 + q_1q_2 + q_1q_4 + q_3q_4 + 1}{q_2q_3q_4 + q_2 + q_4}$$

The polynomial appearing on the numerator and denominator are the continuant polynomials.

Very long history...Mainly examples

1. Rafael Bombelli (1530) presented $\sqrt{13}$ as periodic continued fractions.

2. Pietro Cataldi (1548-1626) presented $\sqrt{18}$ as periodic continued fractions.

3. John Wallis (1616-1703) in his book Arithemetica Infinitorium (1655), studied the continued fractions, Euler laid down much of the modern theory (1737). The non commutative settings started as early as Hamilton followed by Wedderburn, P.M. Cohn...

Definition 1.1. Let $X = \{t_1, t_2, \dots\}$ be a countable set of noncommuting variables, and let $\mathbb{Z}\langle X\rangle$ be the free $\mathbb{Z}$-algebra generated by X. We define the n-th *right continuant polynomials*

$$p_n(t_1, \dots, t_n) \in \mathbb{Z}\langle t_1, \dots, t_n\rangle \subseteq \mathbb{Z}\langle X\rangle$$

by $p_0 = 1$, $p_1(t_1) = t_1$, and inductively for $i \geq 2$ by

$$p_i(t_1, \dots, t_i) = p_{i-1}(t_1, \dots, t_{i-1})\, t_i + p_{i-2}(t_1, \dots, t_{i-2}).$$

the first continuants p_n. They are

$$\begin{aligned} &p_0 = 1, \quad p_1 = t_1, \quad p_2 = t_1t_2 + 1, \quad p_3 = t_1t_2t_3 + t_1 + t_3, \\ &p_4 = t_1t_2t_3t_4 + t_1t_2 + t_1t_4 + t_3t_4 + 1, \\ &p_5 = t_1t_2t_3t_4t_5 + t_1t_2t_3 + t_1t_2t_5 + t_1t_4t_5 + t_3t_4t_5 + t_1 + t_3 + t_5 \end{aligned}$$

They appear, for instance, in:

- Continued fractions
- Generating subgroups of matrices for $M_2(R)$ (P.M. Cohn and below).
- Characterizations of comaximal relations in certain rings (P.M. Cohn).
- Characterizations of Euclidean pairs and quasi Euclidean rings (Alahmadi, Jain, Lam, L.).

For $r \in R$ we denote by $P(r)$ the invertible matrix $P(r) := \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix}$.

Theorem 1.2.

a) $p_n(t_1, \dots, t_n) = t_1 p_{n-1}(t_2, \dots, t_n) + p_{n-2}(t_3, \dots, t_n)$.

b)

$$\mathcal{P}_n := P(t_1)P(t_2)\cdots P(t_n) = \begin{pmatrix} p_n(t_1, \dots, t_n) & p_{n-1}(t_1, \dots, t_{n-1}) \\ p_{n-1}(t_2, \dots, t_n) & p_{n-2}(t_2, \dots, t_{n-1}) \end{pmatrix}$$

.

c) $\mathcal{P}_n^{-1} = (-1)^n \begin{pmatrix} p(t_{n-1},\dots,t_2) & -p(t_{n-1},\dots,t_1) \\ -p(t_n,\dots,t_2) & p(t_n,\dots,t_1) \end{pmatrix}$

Remark that

- p_{2n} is a sum of monomials with an even number of factors.
- p_{2n+1} is a sum of monomials with an odd number of factors.

Put $x_n = t_{2n-1},\ y_n = t_{2n}$ and $G_n = p_{2n},\ H_n = p_{2n-1}$.
So G_n is a polynomial in the indeterminates $x_1, y_1, \dots, x_n, y_n$, and H_n is a polynomial in the indeterminates $x_1, y_1, \dots, x_{n-1}, x_n$.
We have:

$$G_0 = 1, \quad G_1 = x_1y_1 + 1, \quad G_2 = x_1y_1x_2y_2 + x_1y_1 + x_1y_2 + x_2y_2 + 1,$$
$$G_3 = x_1y_1x_2y_2x_3y_3 + x_1y_1x_2y_2 + x_1y_1x_2y_3 + x_1y_1x_3y_3 + \\ + x_1y_2x_3y_3 + x_2y_2x_3y_3 + x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 + 1$$

and

$$H_0 = 0, \quad H_1 = x_1, \quad H_2 = x_1y_1x_2 + x_1 + x_2,$$
$$H_3 = x_1y_1x_2y_2x_3 + x_1y_1x_2 + x_1y_1x_3 + x_1y_2x_3 + x_2y_2x_3 + x_1 + x_2 + x_3.$$

From

$$P(x_i)P(y_i) = \begin{pmatrix} x_iy_i + 1 & x_i \\ y_i & 1 \end{pmatrix},$$

it follows that
(1.I)
$$\begin{pmatrix} x_1y_1 + 1 & x_1 \\ y_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} x_ny_n + 1 & x_n \\ y_n & 1 \end{pmatrix} = \\ \begin{pmatrix} G_n(x_1, y_1, \dots, x_n, y_n) & H_n(x_1, y_1 \dots, y_{n-1}, x_n) \\ H_n(y_1, x_2, \dots, x_n, y_n) & G_{n-1}(y_1, x_2, \dots, y_{n-1}, x_n) \end{pmatrix}.$$

The recursion formulae for p_n translate into the following ones between the G_n, H_n, x_n and y_n:

(1.II) $$G_{n+1} = H_{n+1}y_{n+1} + G_n \quad \text{and} \quad H_{n+1} = G_nx_{n+1} + H_n.$$

Using the associativity of the product of matrices we obtain:

(1.III) $$G_n(x_1, \dots, y_n) = G_k(x_1, \dots, x_{k-1}, y_k)G_{n-k}(x_{k+1}, \dots, y_n) + \\ H_k(x_1, \dots, y_{k-1}, x_k)H_{n-k}(y_{k+1}, x_{k+2} \dots, y_n)$$

and

(1.IV)

$$H_n(x_1, y_1, \dots, x_n) = G_k(x_1, \dots, y_k)H_{n-k}(x_k, y-k, \dots, x_n) + \\ H_k(x_1, y_1, \dots, x_k)G_{n-k-1}(y_{n-k}, \dots, y_{n-1}, x_n)$$

Facts:

- All the polynomials p_n, H_n and G_n are sums of monomials with all the coefficients equal to 1.
- From the defining relations it is easily seen that each G_n is a sum of monomials of all possible even degrees $\leq 2n$ and each H_n is a sum of monomials of all possible odd degrees $\leq 2n-1$.
- Also, the number m_n of monomials in p_n, which is clearly equal to $p_n(1,1,\dots,1)$, satisfies the relations $m_0 = 1, m_1 = 1, m_2 = 2, m_n = m_{n-1} + m_{n-2}$, hence $m_n = F_{n+1}$, the $(n+1)$-th Fibonacci number, defined by $F_0 = 0,\ F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
- The monomials in H_n and G_n do not contain consecutive letters with descending indexes.
- $\frac{\partial p_{n+1}(t_1,\dots,t_{n+1})}{\partial t_{n+1}} = p_n; \qquad \frac{\partial G_n(x_1,y_1,\dots,x_n,y_n)}{\partial y_n} = H_n(x_1, y_1, \dots, x_{n-1}, x_n)$

2 Continuants and groups of matrices

Let R be any ring, $GL_2(R)$ the group of all invertible 2×2-matrices $E_2(R)$ be the *elementary group*, that is, the subgroup of $GL_2(R)$ generated by matrices $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, where x and y range in R.

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$$

An arbitrary element of $E_2(R)$ is a product of finitely many elements of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} xy+1 & x \\ y & 1 \end{pmatrix}.$$

These are exactly the factors on the left in the equation (1.I). Thus an arbitrary element of $E_2(R)$ is a matrix of the form

$$\begin{pmatrix} G_n(x_1, y_1, \dots, x_n, y_n) & H_n(x_1, y_1 \dots, y_{n-1}, x_n) \\ H_n(y_1, x_2, \dots, x_n, y_n) & G_{n-1}(y_1, x_2, \dots, y_{n-1}, x_n) \end{pmatrix},$$

with $x_1, y_1, \dots, x_n, y_n \in R$.

Let G be the subsemigroup of the multiplicative semigroup $M_2(R)$ generated by all matrices of type

$$P(x) := \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

where x ranges in R. As Cohn proved in 1966, the semigroup G, set of all products $P(x_1) \cdots P(x_n)$ with $n \geq 1$ and $x_1, \dots, x_n \in R$, is a group, because $P(0)^2$ is the identity of $GL_2(R)$ and $P(x)^{-1} = P(0)P(-x)P(0)$.

Theorem 2.1. *For any ring R, exactly one of the following two conditions holds:*

(a) *Either $G = E_2(R)$, or*

(b) *The group G is the semidirect product $E_2(R) \rtimes C$ of the group $E_2(R)$ and the cyclic group C of order 2 generated by the involution $P(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

The action of $P(0)$ on $E_2(R)$ is given by

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto P(0)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}P(0) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \mapsto P(0)\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}P(0) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

3 Leapfrogs and graphs

First leapfrog construction

0) The first term of p_n is $t_1 t_2 \cdots t_n$.
1) The next terms are obtained by erasing two consecutive indeterminates (the frog jumps over them) from $t_1 t_2 \cdots t_n$ to get the sum: $t_3 t_4 \cdots t_n + t_1 t_4 t_5 \cdots t_n + t_1 t_2 t_5 \cdots t_n + \ldots$.
2) We erase 2 pairs of consecutive indeterminates (2 jumps) and get the terms

$$\sum_{1\le i_1<i_2-1\le n} t_1 \cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}} \cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}} \cdots t_n$$

.

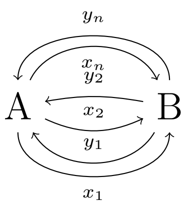
3) We then continue adding terms corresponding to 3 leaps, 4 leaps, and so on. Finally, we can write

$$p_n(t_1, \ldots, t_n) = \sum_{i_1, i_2, \ldots, i_j} t_1 \cdots \widehat{t_{i_1}}\widehat{t_{i_1+1}} \cdots \widehat{t_{i_2}}\widehat{t_{i_2+1}} \cdots \widehat{t_{i_j}}\widehat{t_{i_j+1}} \cdots t_n$$

where $1 \le j \le \lfloor n/2 \rfloor$ and $i_j + 1 < i_{j+1}$ for every j,

Graph and second leapfrog construction

Consider the directed graph (quiver) Γ_n with two vertices A and B:



Thus Γ_n has $2n$ arrows, of which n goes from A to B and are indexed by the indeterminates x_i, and n from B to A indexed by the indeterminates y_i.

The facts mentioned earlier about the polynomials G_n and H_n motivate the definition of the ideal I in the construction below.
Let k be a field, consider the quiver algebra $k\Gamma_n$ and the ideal I of $k\Gamma_n$ generated by all paths $x_i y_j \colon A \xrightarrow{x_i} B \xrightarrow{y_j} A$ with $i > j$ and all paths $y_i x_j \colon B \xrightarrow{y_i} A \xrightarrow{x_j} B$ with $i \ge j$.

Theorem 3.1. *Let $R = k\Gamma_n/I$.*

1) The k-algebra R is finite dimensional.

2) $J(R)$ is a nilpotent ideal that contains all nilpotent elements of R.

3) $R = R_0 \oplus R_1$ is 2-graded, where R_0 corresponds to the paths of even length and R_1 to the paths of odd length.

4) The images of the polynomials G_n in R are in R_0 and the images of the polynomials H_n are in R_1.

5)

$$H_n = \left(1 - \sum_{1\le i\le j\le n} x_i y_j\right)^{-1} \left(\sum_{i=1} x_i\right) \quad \text{and} \quad G_n = \left(1 - \sum_{1\le i\le j\le n} x_i y_j\right)^{-1}$$

for every $n \geq 0$.

4 Generalized Fibonacci Polynomials and tiling of strips

Definition 4.1. The polynomials $f_n \in \mathbb{Z}\langle x_1, y_1, x_2, y_2, \ldots, \rangle$ are defined by the recursion formulae:
(4.I)

$$f_{-1} = 0, \quad f_0 = 1,$$
$$f_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = f_{n-1}(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})x_n + \\ + f_{n-2}(x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2})y_n.$$

The first of these polynomials f_n are

$$f_0 = 1, \qquad f_1 = x_1, \qquad f_2 = x_1x_2 + y_2,$$
$$f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3,$$
$$f_4 = x_1x_2x_3x_4 + x_1x_2y_4 + x_1y_3x_4 + y_2x_3x_4 + y_2y_4,$$
$$f_5 = x_1x_2x_3x_4x_5 + x_1x_2x_3y_5 + x_1x_2y_4x_5 + x_1y_3x_4x_5 + \\ + x_1y_3y_5 + y_2x_3x_4x_5 + y_2x_3y_5 + y_2y_4x_5, \ldots$$

- The number of monomials in each f_n is the $(n+1)$-th Fibonacci number F_{n+1}.

- When we specialize all the indeterminates y_i to 1, we get back the continuant polynomials i.e. $f_n(x_1,\dots,x_n,1,\dots,1)=p_n(x_,\dots,x_n)$.
- If we specialize further: $f_n(x,\dots,x,1,1,\dots,1)=F_n(x)$, i.e. we get the commutative Fibonacci polynomials.
- The polynomials f_n are homogeneous of degree n if we give the x_i degree one and the y_i degree 2.
- Notice that the indeterminate y_1 does not appear in any polynomial $f_n(x_1,\dots,x_n,y_1,\dots,y_n)$.

Theorem 4.2. *1.* $f_n(2,2,\dots,2,-1,-1,\dots,-1)=n$

2. $f_n(x+1,x+1,\dots,x+1,-x,-x,\dots,-x)=1+x+x^2+\dots+x^{n-1}$.

3. We have:

$$\mathcal{F}_n := \begin{pmatrix} x_1 & 1 \\ y_1 & 0 \end{pmatrix}\cdots\begin{pmatrix} x_n & 1 \\ y_n & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} f_n(x_1,\dots,x_n,y_1,\dots,y_n) & f_{n-1}(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1}) \\ y_1f_{n-1}(x_2,\dots,x_n,y_2,\dots,y_n) & y_1f_{n-2}(x_2,\dots,x_{n-1},y_2,\dots,y_{n-1}) \end{pmatrix}.$$

4.

$$\mathcal{F}_n = \begin{pmatrix} f_k(x_1,\dots,y_k) & f_{k-1}(x_1,\dots,y_{k-1}) \\ y_1f_{k-1}(x_2,\dots,y_k) & y_1f_{k-2}(x_2,\dots,y_{k-1}) \end{pmatrix}\cdot$$
$$\cdot\begin{pmatrix} f_{n-k}(x_{k+1},\dots,y_n) & f_{n-k-1}(x_{k+1},\dots,y_{n-1}) \\ y_{k+1}f_{n-k-1}(x_{k+2},\dots,y_n) & y_{k+1}f_{n-k-2}(x_{k+2},\dots,y_{n-1}) \end{pmatrix}$$

5. $f_n(x_1,\ \dots,\ x_n,\ y_1,\ x_1x_2,\ x_2x_3,\ x_3x_4,\ \dots,\ x_{n-1}x_n)=F_{n+1}x_1x_2\dots x_n$.

6. $f_n(x_1,\dots,y_n)=f_k(x_1,\dots,y_k)f_{n-k}(x_{k+1},\dots,y_n)+$
$+f_{k-1}(x_1,\dots,y_{k-1})y_{k+1}f_{n-k-1}(x_{k+2},\dots,y_n)$

7. $f_n(x_1,\dots,x_n,y_1,\dots,y_n)=$
$=x_1f_{n-1}(x_2,\dots,x_n,y_2,\dots,y_n)+y_2f_{n-2}(x_3,\dots,x_n,y_3,\dots,y_n)$.

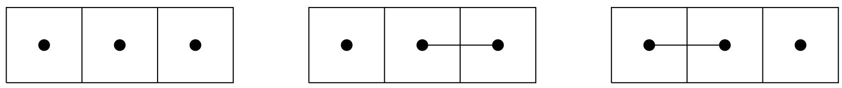
8. $f_n(x_1,x_2,\dots,y_n)=$
$=f_{k+1}(x_1,\dots,x_k,f_{n-k}(x_{k+1},\dots,y_n),y_1,\dots,y_k,f_{n-k-1}(x_{k+2},\dots,y_n))$.

9. $\frac{\partial f_n(x_1,\dots,y_n)}{\partial x_k}=f_{k-1}(x_1,\dots,y_{k-1})f_{n-k}(x_{k+1},\dots,y_n)$, for $1\le k\le n$.
$\frac{\partial f_n(x_1,\dots,y_n)}{\partial y_k}=f_{k-2}(x_1,\dots,y_{k-2})f_{n-k}(x_{k+1},\dots,y_n)$, for $2\le k\le n$.

It is also possible to describe the generalized Fibonacci polynomials via leapfrog constructions and a path algebra can also be defined based on this definition.

Definition 4.3. A *linear tiling* of a row of squares (a $1 \times n$ strip of square cells) is a covering of the strip of squares with squares and dominos (which cover two squares).

For instance, the polynomial $f_3 = x_1x_2x_3 + x_1y_3 + y_2x_3$ parametrizes the set of the three linear tilings



of a row of three squares. Here x_i denotes the i-th square and y_i denotes the domino that covers the $(i-1)$-th and the i-th square (the domino that "ends on the i-th square".) The Fibonacci number F_n represents the number of tilings of a strip of length n using length 1 squares and length 2 dominos.

5 General pattern and permanents.

Now consider the following family of polynomials g_n, with $n \geq 0$. To define them, we need countably many non-commutative indeterminates x_{ij}, where $1 \leq i \leq j$. Set $g_0 = 1$ and

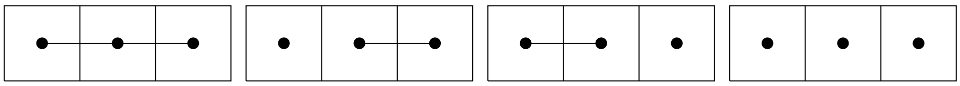
$$g_n = \sum_{i=1}^{n} g_{i-1}x_{in}, \quad \text{for} \quad n \geq 1. \tag{5.I}$$

For instance, the first polynomials g_n are

$$g_1 = x_{11}, \quad g_2 = x_{12} + x_{11}x_{22}, \quad g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33},$$
$$g_4 = x_{14} + x_{11}x_{24} + x_{12}x_{34} + x_{11}x_{22}x_{34} + x_{13}x_{44} + x_{11}x_{23}x_{44} + $$
$$+ x_{12}x_{33}x_{44} + x_{11}x_{22}x_{33}x_{44}.$$

For every $n \geq 1$, the polynomial $g_n \in \mathbb{Z}\langle x_{ij} \,|\, 1 \leq i \leq j \leq n \rangle$. The polynomial g_n is a sum of monic monomials that parametrize all linear tilings of a strip of n square cells, that is, all coverings of the strip of squares with rectangles of any length $1, 2, \ldots, n$. The indeterminate x_{ij} indicates the rectangle of length $j - i + 1$ that starts from the i-th square and ends covering the j-th square.

For instance, $g_3 = x_{13} + x_{11}x_{23} + x_{12}x_{33} + x_{11}x_{22}x_{33}$ and, correspondingly, the tilings of a strip of three squares are



We can get back the polynomials p_n and f_n by different specializations.

We have:

$$(g_1, \ldots, g_n) = (g_0, \ldots, g_{n-1}) \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ 0 & x_{22} & \ldots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & x_{nn} \end{pmatrix}$$

Since, for $1 \leq l \leq n$, a tiling of a strip of length n is obtained by a tile of length l followed by a tiling of length $n - l$, the following formula, where we have specified explicitly the indeterminates ("the tiles") for each polynomial, is easy to get:

$$g_n(x_{ij}; 1 \leq i \leq j \leq n) = \sum_{l=1}^{n} x_{1l} g_{n-l}(x_{l+i,l+j}; 1 \leq i \leq j \leq n - l)$$

R a ring, define a mapping $\mathrm{perm}\colon M_n(R) \to R$ setting, for every matrix $A = (a_{i,j})_{i,j} \in M_n(R)$,

$$\mathrm{perm}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} \ldots a_{n,\sigma(n)}.$$

If $A_{i,j}$ denotes the $(n-1) \times (n-1)$-matrix that results from A removing the i-th row and the j-th column, then $\mathrm{perm}(A) := \sum_{j=1}^{n} a_{1,j} \mathrm{perm}(A_{1,j}) = \sum_{j=1}^{n} \mathrm{perm}(A_{n,j}) a_{n,j}$ (it is possible to expand our permanent along the first row or the last row only).

Theorem 5.1. *For every $n \geq 1$, we have:*

$$g_n(x_{ij}) = \mathrm{perm}(A_n) = \mathrm{perm}(A_n^t),$$

where

$$A_n = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 1 & x_{22} & x_{23} & \dots & x_{2n} \\ 0 & 1 & x_{33} & \ddots & x_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & x_{nn} \end{pmatrix}$$

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THANK YOU !