Celebrating Walter Ferrer 70^{th} birthday

Commutatively Closed Sets in Rings

Joint work with Mona Abdi and Dilshad Alghazzawi

Colloquium on Algebras and Representations
Quantum 19

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Notations and Definitions

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Remark

The set of commutatively closed subsets of R defines a topology on R.

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- *If φ is an isomorphism then for any $X \subseteq R$, we have $\overline{\varphi(X)} = \varphi(\overline{X})$.*

Theorem

Let S be a subset of a ring R. $r(S)$ ($l(S)$) denotes the right (left) annihilator of S.

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6. *If a is a right (or left) invertible element then a and all of its factors are units.*

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In particular, a is strongly regular and the idempotent ax is central.

Definition

For $a \in R$, $C(a) := \overline{\{a\}}$.
$C(a)$ is a graph: $x, y \in C(a)$ are connected if $x \sim_1 y$.
We can also define a distance in $C(a)$: $d(x, y) = n$ when $x \in \{y\}_n$ but $x \notin \{y\}_{n-1}$.

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2. *If $a \sim_n b$, then $a - b$ is a finite sum of additive commutators.*

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1. *For any $n \geq 1$ and $a, b \in R$, we have $a \sim_n b$ if and only if there exist $x_1, x_2, \ldots, x_n \in R$ and $y_1, y_2, \ldots, y_n \in R$ such that $a = x_1y_1, y_1x_1 = x_2y_2, y_2x_2 = x_3y_3, \ldots, y_nx_n = b$.*
2. *If $a \sim_n b$, then $a - b$ is a finite sum of additive commutators.*
3. *If $a \sim_n b$ then there exist $x, y \in R$ such that $ax = xb$ and $ya = by$. We then have $yx \in Z(b)$ and $xy \in Z(a)$, where, for $x \in R$, $Z(x)$ denotes the centralizer of x.*

Examples

1. Consider the algebra $k\langle X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_n\rangle/I$ where $I = Idl < Y_1X_1 - X_2Y_2, Y_2X_2 - X_3Y_3, \ldots Y_{n-1}X_{n-1} - X_nY_n >$. We write x_i, y_i for $X_i + I$, $Y_i + I$. In $\overline{x_1y_1}$. We have that $d(x_1y_1, x_ny_n) = n$.

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2. Let $R = K\langle x, y\rangle$ be free K-algebra. Since $x + yx^l = (1 + yx^{l-1})x \sim_1 x(1 + yx^{l-1}) = (1 + xyx_{l-2})x \sim_1 x(1 + xy^{l-2}) = (1 + x^2yx^{l-3})x \sim_1 \cdots \sim_1 x(1 + x^{l-1}y)$, so $d(x + yx^l, x + x^ly) \leq l$. In fact $d(x + yx^l, x + x^ly) = l$.

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4. Let $a \in k$ where k is a field and $\sigma \in Aut(k)$, $n \in \mathbb{N}$. In $R = k[t; \sigma]$ we have $t^n - 1$ is C.C.

Periodic elements

Definition

An element $x \in R$ is a-periodic ($a \in Z(R)$) if there exist nonzero natural numbers $n, m \in \mathbb{N}$, $n \neq m$, such that $x^n = ax^m$. If $a = 1$ we just say that x is periodic. The 0-periodic elements are the nilpotent elements.

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Proposition

If $a \in Z(R)$ and $b \sim a$ then b is a-periodic. The set of a-periodic elements is commutatively closed. The class $\overline{\{1\}}$ is contained in the class of periodic elements.

Example

Let $R = M_2(\mathbb{F}_2)$. We describe the different classes:

- $\overline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$

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Matrix rings over a field

Lemma

Let k be a field and let $A, B \in R = M_n(k)$ be two square matrices such that $A \sim B$, then the characteristic polynomials of A and B are equal.

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Proposition

Let k be a commutative field and $n \in \mathbb{N}$, the class $\overline{\{0\}}$ in $M_n(k)$ is the set of nilpotent matrices, moreover $diam(\overline{\{0\}}) = n - 1$.

Proposition

Let $UT_n(R)$ denote the ring of upper triangular matrices having 1's on the diagonal. Then $\overline{\{0\}}$ is the set of nilpotent elements.

Proposition

Let R be a field that is not Dedekind finite, then $diam(R) = \infty$.

Proof.

Sketch: The proof is based on the fact that if $ab = 1$ but $ba \neq 1$ then, for $i, j \in \mathbb{N}$

$$e_{ij} := b^i(1 - ba)a^j \text{ are such that } e_{ij}ekl = \delta_{jk}e_{i,l}$$

Then for any $n \in \mathbb{N}$ consider $a_n := e_{12} + e_{23} + \cdots + e_{n-1,n}$ this element is nilpotent and $d(a_n, 0) = n - 1$. □

2-primal rings

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Proposition

(a) *Let R be a ring such that $\{0\}_1$ is contained in the center $Z(R)$. Then R is 2-primal.*

(b) *The prime radical $P(R)$ of a ring R is commutatively closed if and only if R is 2-primal.*

Example

The converse of (a) in the above proposition is not true. If k is a field, the ring $R = k[x][t;\sigma]/(t^2)$, where σ is the k-algebra map defined by $\sigma(x) = 0$, is easily seen to be 2-primal but $xt + (t^2) \in \{0\}_1$ and is not central.

Definition

For a subset $E \subseteq R$ the commutative depth of E, denoted $Cd(E)$, is defined to be the smallest $l \in \mathbb{N}$ such that $E_l = E_{l+1}$ when such an l exists. If no such integer l exists then we put $Cd(E) = \infty$.

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1. E is commutatively closed if and only if $Cd(E) = 0$
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Examples

1. E is commutatively closed if and only if $Cd(E) = 0$
2. If R is the upper triangular matrix ring over $\mathbb{F}_2$ then $Cd(R) = 1$.
3. For subsets E and F of a ring R, we have $Cd(E \cup F) \leq max\{Cd(E), Cd(F)\}$

Open problems

Question 1: Characterize the rings R such that $diam(R) = 1$.

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Question 2: Let k be a field, compute $Diam(M_n(k))$.

THANK YOU !
Happy birthday Walter !