Title

Singular matrices as products of idempotent matrices

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Joint work with A. Alahmadi, S.K. Jain, T.Y. Lam.

Plan

A History and introduction.

Plan

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B First examples and remarks.

Plan

A History and introduction.

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C Singular matrices over division rings.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

D Singular matrices over local rings.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

D Singular matrices over local rings.

E Hermite domains.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

D Singular matrices over local rings.

E Hermite domains.

F Quasi-Euclidean rings.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

D Singular matrices over local rings.

E Hermite domains.

F Quasi-Euclidean rings.

G GE-rings.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

D Singular matrices over local rings.

E Hermite domains.

F Quasi-Euclidean rings.

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H Nonnegative singular matrices.

Plan

A History and introduction.

B First examples and remarks.

C Singular matrices over division rings.

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E Hermite domains.

F Quasi-Euclidean rings.

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I Hannah and O'Meara's works.

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6. Number of idempotents needed (Ballantine, Laffey, Laudau X...)

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First examples and remarks, I

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Let R be any ring and let $a, b, c \in R$. Then

(a) $$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

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(d) *with* $b \in U(R)$, $\begin{pmatrix} a & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} b(b^{-1}a) & b \\ 0 & 0 \end{pmatrix}$ *can be factorized as in (c).*

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Lemma

1. *The following decompositions appear often in the proofs:*
   - $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$, *where C is a column.*

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   - $\begin{pmatrix} B & 0 \\ R & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}$, *where R is a row.*

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   - $\begin{pmatrix} B & 0 \\ R & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ R & 0 \end{pmatrix}$, *where R is a row.*
   - *If $B \in GL_{n-1}(R)$ then* $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} & B^{-1}C \\ 0 & 0 \end{pmatrix}$
2. *If R is a right Bézout domain then any singular matrix is similar to a matrix having its last row zero.*

First examples and remarks, III

Proposition {Alahmadi,Jain, L.}

The following matrices are always product of idempotent matrices

- Singular (0,1) matrices,

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Proposition {Alahmadi,Jain, L.}

The following matrices are always product of idempotent matrices

- Singular (0,1) matrices,
- Strictly upper triangular matrices,
- Quasi permutation matrices,
- Quasi elementary matrices.

Division rings

Theorem {Laffey}

A singular matrix with coefficients in a division ring is always a product of idempotent matrices.

Steps of the proof:

- Reduce to a matrix of the form $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$
- If $n = 2$, use the decomposition from the introduction.
- If $n > 2$ and B is singular then by induction it is a product of idempotents.
- If B is invertible we can write

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ I_{n-1,n-1} & 0 \end{pmatrix} \begin{pmatrix} I_{n-1,n-1} & B^{-1}C \\ 0 & 0 \end{pmatrix}$$

Steps of the proof for division rings, II

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B' & D \\ 0 & 0 \end{pmatrix}$$

where $B' \in M_{n-1,n-1}(D)$ has its first column zero and D is a column vector. This means that B' is singular and the induction hypothesis implies that B' is in fact a product of idempotents, say $B' = E_1 \dots E_r$ where $E_i^2 = E_i$ for any $1 \leq i \leq r$. We then have

$$\begin{pmatrix} B' & D \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1,n-1} & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} E_r & 0 \\ 0 & 1 \end{pmatrix}.$$

local rings

Theorem {Jain, L.}

Let R be a local ring. Suppose that every 2×2 matrix over R having nonzero right or left annihilator is product of idempotents. Then R must be a valuation domain.

Definition

A ring is a right Hermite ring if its f.g. right stably free modules are free.

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Lemma {Jain, L.}

A singular matrix with coefficients in a right Hermite domain is similar to a matrix with its last row equal to zero.

Assuming moreover that the ring is a GE-ring (i.e. every invertible matrix is a product of elementary matrices) we "easily" get that

Theorem {Ruitenburg and Jain, Lam, L, }

If R is a GE right Hermite domain then any singular matrix with coefficients in R is a product of idempotent matrices.

quasi Euclidean rings

Definitions

1. A pair $(a, b) \in R^2$ is a *right Euclidean pair* if there exist elements $(q_1, r_1), \ldots, (q_{n+1}, r_{n+1}) \in R^2$ (for some $n \geq 0$) such that $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

$$(*) \qquad r_{i-1} = r_i q_{i+1} + r_{i+1} \text{ for } 1 < i \leq n, \text{ with } r_{n+1} = 0.$$

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The notion of a left Euclidean pair is defined similarly.

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   The notion of a left Euclidean pair is defined similarly.
   A ring R is right quasi-euclidean if every pair (a, b) is a right Euclidean pair.
2. A ring R is of stable range 1 if for any $(a, b) \in R^2$ such that $aR + bR = R$ there exists $x \in R$ such that $a + bx$ is invertible.

Suppose (a, b) is a right Euclidean pair with $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, and

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- In matrix form we get the following

$$(a, b) = (r_n, 0)\, P(q_{n+1}) \cdots P(q_1).$$

where $P(q)$ is the invertible matrix $\begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$.

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- Let us develop the right handside product of matrices:

$$\begin{pmatrix} q_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_1 q_2 + 1 & q_1 \\ q_2 & 1 \end{pmatrix}$$

  Continuing this process we arrive at the contnuant polynomials but...this is another story!

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(D) For any $a, b \in R$, $(a, b) = (r, 0)\,Q$ for some $r \in R$ and $Q \in \mathrm{GE}_2(R)$.

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(E) For any $a, b \in R$, $(a, b) = (r, 0)\, Q$ for some $r \in R$ and $Q \in \mathrm{E}_2(R)$.

More properties

Theorem {Alahmadi, Jain, Lam,L.}

(a) Any unit regular ring is (right and left) quasi-Euclidean.

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(d) Let R be a right Bézout ring and I be any ideal contained in the Jacobson radical $J(R)$. R/I is right quasi-Euclidean iff R is right quasi Euclidean.

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(f) If R and S are two right quasi-Euclidan rings then $R \times S$ is right quasi-Euclidean.

Theorem {Alahmadi, Jain, Lam,L.}

A domain R is right quasi-Euclidean if and only if R is a projective-free GE_2-ring such that every matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ is a product of idempotents in $\mathbb{M}_2(R)$.

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Theorem {Alahmadi, Jain, Lam,L.}

Let $A \in M_n(R)$ where R is a right and left quasi Euclidean ring. Then:

1. $l(A) \neq 0$ if and only of $r(A) \neq 0$.
2. If $l(A) \neq 0$ then A is a product of idempotent matrices.

The proof of (2) in the above theorem follows the line of the one given by Laffey given for classical commutative Euclidean domains.

The importance of the GE property for decomposing matrices into idempotents can be easily seen from the following somewhat technical result:

Lemma

If R is a GE ring and $B \in GL_n(R)$, then the matrix

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$$

is a product of idempotent matrices.

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Salce and Zanardo analyzed the relation between the two decompositions. They studied the case of commutative domains but their results were generalized to a noncommutative domains by Facchini and Leroy. To present the latter result we need to introduce a few notions.

Definitions

- Let A, B, C be three right R-modules and $\alpha\colon A \to B$, $\beta\colon B \to C$ be homomorphisms. We say that the pair (α, β) is a *consecutive pair* if $im(\alpha) \oplus \ker(\beta) = B$

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- We say that a ring R is *right n-regular* if for every $n \times n$ invertible matrix $M = (b_{ij}) \in M_n(R)$ there exists some $i, j = 1, 2, \ldots, n$ such that $r(b_{ij}) = 0$.

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- Let r, n be integers, $0 \leq r \leq n$. For a ring R we define $\mathcal{F}_{n,r} := \{ A \subseteq^{\oplus} R_R^n \mid A \cong R_R^r \text{ and } R_R^n / A \cong R_R^{n-r} \}$

Theorem {Facchini, L.}

R a ring with IBN and $n \geq 1$. Suppose R is m-right regular for every $m \leq n$ and that for any two decompositions of $R^n = A \oplus X = Y \oplus B$ with A, B free right of ranks, respectively, $n-1, 1$, the submodules X, Y are free right R-modules. T.F.A.E.:

$(\mathrm{HI}_{n,1})$ For every free direct summands $A \subseteq^{\oplus} R_R^n$ and $B \subseteq^{\oplus} R_R^n$, with A, B free R-modules of rank $n-1, 1$ respectively, there exists an endomorphism β of R_R^n with $\mathrm{im}(\beta) = A$ and $\ker(\beta) = B$, which is a product $\beta = \varepsilon_1 \ldots \varepsilon_k$ of consecutive idempotent $(\mathcal{F}_{n,n-1}, \mathcal{F}_{n,1})$-endomorphisms.

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(GE_n) Every invertible $n \times n$ matrix is a product of elementary matrices.

Question and particular matrices

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Main Question : Can real **nonnegative** singular matrices be decomposed into product of **nonnegative** idempotents ?

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Lemma

Particular matrices

(a) *If $B \in M_{n\times n}(\mathbb{R}^+)$ is an $n \times n$ matrix which is a product of nonnegative idempotents, then the same is true for the matrix* $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$ *where $C \in M_{n\times 1}(\mathbb{R}^+)$ and the other blocks are of appropriate sizes.*

(b) *If $A \in M_n(\mathbb{R})$ (resp. $A \in M_n(\mathbb{R}^+)$), $n \geq 3$, has all its i^{th} rows and columns zero whenever $i \geq 3$, then A is a product of*

Rank one

Proposition {Alahmadi,Jain, Sathaye, L.}

Let $A \in M_n(\mathbb{R}^+)$, $n > 1$, be a nonnegative matrix of rank 1. Then A is a product of nonnegative idempotent matrices.

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Remark (A.,J.L.,S.)

It can be shown that in fact rank 1 nonnegative matrices can be decomposed into a product of three nonnegative idempotent matrices.

Rank two

Theorem {A.,J.,L.}

Let $A \in M_n(\mathbb{R}^+)$, $n > 2$, be a nonnegative singular matrix of rank 2. Then A is a product of nonnegative idempotent matrices.

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The proof is based on the fact that when the rank is two the rows of the matrix can be expressed as nonnegative linear combinations of two generating rows.

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This theorem is no longer valid for matrices with rank > 2.

counter-example

For singular nonnegative matrices of higher rank the decomposition does not necessarily exist:

Example

$$A_\alpha := \begin{pmatrix} \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha \end{pmatrix}, \quad \textit{where } \alpha \in \mathbb{R}^+,\ \alpha \neq 0.$$

If $A_\alpha = E_1 \dots E_n$ is such that $E_i^2 = E_i \in M_n(\mathbb{R}^+)$ then $A_\alpha = A_\alpha E_n$ and a direct computation shows that $E_n = Id$.. Remark that $A_{\frac{1}{2}}$ is a positive doubly stochastic matrix.

Nilpotent matrices

Proposition {Jain, Goel}

If A is Nonnegative nilpotent there exists a permutation matrix such that PAP^t is an upper triangular nonnegative matrix.

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Corollary

Nonnegative nilpotent matrices are product of nonnegative idempotent matrices.

Hannah and O'Meara

Hannah and O'Meara published several interesting results on the decomposition of nonunit elements of a regular ring into idempotents.

Theorem

If an element a of a regular ring R is a product of k idempotents then $(1-a)R \leq k.rann(a)$.

Corollary

An element a in a unit regular ring is a product of idempotents if and only if $R.rann(a) = R(1-a)R$.

Hannah and O'Meara also proved the following remarkable result:

Theorem

Let R be one of the following rings: (i) unit regular, (ii) right continuous, or (iii) a factor ring of a right self-injective ring. Then a is a product of idempotents if and only if

$$R.rann(a) = R(1-a)R = lann(a).R$$

A ring R is separative if for all finitely generated projective modules, A, B

$$A \oplus A \simeq A \oplus B \simeq B \oplus B \quad \text{implies} \quad A \simeq B$$

Equivalently, $2A \cong 2B$ implies $A \cong B$

Theorem

Let R be a regular ring. Then the separativity of R is equivalent to the fact that an element is a product of idempotents if and only of $R.rann(a) = R(1-a)R = lann(a).R$

It is worthy to mention that no example of a regular ring that is not separative is known. This is certainly one of the most important open problems in regular rings.

Thank you for your attention.