Mass formulae for self-orthogonal, self-dual and LCD codes over finite commutative chain rings

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1 This is a joint work with Ms. Monika Yadav, Department of Mathematics, IIIT-Delhi.
$\mathcal{R}_e$  a finite commutative chain ring with the nilpotency index $e$

$u$  a generator of the maximal ideal of $\mathcal{R}_e$

$\overline{\mathcal{R}}_e$  $\mathcal{R}_e/\langle u \rangle$, the residue field of $\mathcal{R}_e$
Some preliminaries
Mass formulae
Classification

Linear code over $\mathcal{R}_e$ and its dual code
Mass Formulae for self-orthogonal codes over finite fields

$n$ positive integer

$\mathcal{R}_e^n$ $\mathcal{R}_e$-module consisting of all $n$-tuples over $\mathcal{R}_e$

### Linear code

A linear code $C$ of length $n$ over $\mathcal{R}_e$ is defined as an $\mathcal{R}_e$-submodule of $\mathcal{R}_e^n$.

### Generator matrix for a linear code

A generator matrix for a linear code $C$ is defined as a matrix over $\mathcal{R}_e$ whose rows form a minimal generating set of the code $C$. 
Next for positive integers $k$ and $\ell$, let $M_{k \times \ell}(R_e)$ denote the set of all $k \times \ell$ matrices over $R_e$.

**Theorem [Norton and Sălăgean (2000)]**

Every linear code $C$ of length $n$ over $R_e$ is permutation equivalent to a code with a generator matrix $G$ in the standard form

$$G = \begin{bmatrix}
I_{k_1} & A_{1,1} & A_{1,2} & \cdots & A_{1,e-1} & A_{1,e} \\
0 & uI_{k_2} & uA_{2,2} & \cdots & uA_{2,e-1} & uA_{2,e} \\
0 & 0 & 0 & \cdots & u^{e-2}A_{e-1,e-1} & u^{e-2}A_{e-1,e} \\
0 & 0 & 0 & \cdots & u^{e-1}I_{k_e} & u^{e-1}A_{e,e}
\end{bmatrix}, \quad (1)$$

where the columns of the matrix $G$ are grouped into blocks of sizes $k_1$, $k_2$, $k_3$, $\cdots$, $k_{e-1}$, $k_e$, $k_{e+1} = n - (k_1 + k_2 + \cdots + k_e)$, the matrix $I_{k_i}$ is the $k_i \times k_i$ identity matrix over $R_e$ and the matrix $A_{i,j} \in M_{k_i \times k_j+1}(R_e)$ is considered modulo $u^{j-i+1}$ for $1 \leq i \leq j \leq e$.

A linear code $C$ of length $n$ over $R_e$ is said to be of the type $\{k_1, k_2, k_3, \cdots, k_e\}$ if it is permutation equivalent to a code with a generator matrix $G$ of the form (1).
Euclidean bilinear form

The Euclidean bilinear form is a mapping $\langle \cdot, \cdot \rangle : \mathcal{R}_e^n \times \mathcal{R}_e^n \rightarrow \mathcal{R}_e$, defined as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

for $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathcal{R}_e^n$.

The dual code

The dual code $C^\perp$ of a linear code $C$ of length $n$ over $\mathcal{R}_e$ is defined as

$$C^\perp = \{ y \in \mathcal{R}_e^n \mid \langle x, y \rangle = 0 \text{ for all } x \in C \}.$$

Note that

1. the dual code $C^\perp$ is also a linear code of length $n$ over $\mathcal{R}_e$.
2. if the code $C$ is of the type $\{k_1, k_2, \cdots, k_{e-1}, k_e\}$, then the dual code $C^\perp$ is of the type $\{n - (k_1 + k_2 + \cdots + k_e), k_e, k_{e-1}, \cdots, k_2\}$.
Definition

A linear code $C$ of length $n$ over $\mathbb{R}_e$ is said to be

1. self-orthogonal if it satisfies $C \subseteq C^\perp$.
2. self-dual if it satisfies $C = C^\perp$.
3. linear with complementary dual (LCD) if it satisfies $C \cap C^\perp = \{0\}$. 
For an integer $k$ satisfying $0 \leq k \leq n$ and a prime power $q$, let

$$\sigma_q(n, k)$$

the number of distinct (Euclidean) self-orthogonal codes of length $n$ and dimension $k$ over the finite field $\mathbb{F}_q$

Note that
- $\sigma_q(n, 0) = 1$.
- $\sigma_q(n, k) = 0$ for $k > \frac{n}{2}$.
Theorem [Pless (1968)]

For an integer \( k \) satisfying \( 1 \leq k \leq \frac{n}{2} \) and a prime power \( q \), we have

\[
\sigma_q(n, k) = \begin{cases}
\frac{\prod_{i=0}^{k-1} (q^{n-1-2i} - 1)}{\prod_{j=1}^{k} (q^j - 1)} & \text{if } n \text{ is odd}; \\
\frac{(q^{n-k} - 1) \prod_{i=1}^{k-1} (q^{n-2i} - 1)}{\prod_{j=1}^{k} (q^j - 1)} & \text{if both } n \text{ and } q \text{ are even}; \\
\frac{(q^{n-k} - q^{\frac{n}{2}} - k + q^{\frac{n}{2}} - 1) \prod_{i=1}^{k-1} (q^{n-2i} - 1)}{\prod_{j=1}^{k} (q^j - 1)} & \text{if } n \text{ is even, } q \text{ is odd and } (-1)^{\frac{n}{2}} \text{ is a square in } \mathbb{F}_q; \\
\frac{(q^{n-k} + q^{\frac{n}{2}} - k - q^{\frac{n}{2}} - 1) \prod_{i=1}^{k-1} (q^{n-2i} - 1)}{\prod_{j=1}^{k} (q^j - 1)} & \text{if } n \text{ is even, } q \text{ is odd and } (-1)^{\frac{n}{2}} \text{ is not a square in } \mathbb{F}_q.
\end{cases}
\]
From now on, let

\[k_1, k_2, \cdots, k_{e+1}\] non-negative integers, not all zero

\[n = k_1 + k_2 + \cdots + k_{e+1}\]

For integers \(t, \ell\) satisfying \(2 \leq t \leq \lceil \frac{e+1}{2} \rceil\) and \(1 \leq \ell \leq t - 1\), let us define

\[h_\ell(k_1, k_2, \cdots, k_t) = (k_1 + k_2 + \cdots + k_\ell)(n - (k_1 + k_2 + \cdots, k_{\ell+1}) - 1),\]

\[n_\ell(k_1, k_2, \cdots, k_t) = (k_1 + k_2 + \cdots + k_\ell)(n - (k_1 + k_2 + \cdots, k_{\ell+1}) - 1) + \left((k_1 + k_2 + \cdots + k_{t-\beta}) + (k_1 + k_2 + \cdots + k_t) - (k_1 + k_2 + \cdots + k_{\ell+1})\right)(n - (k_1 + k_2 + \cdots + k_{t-\beta}) - (k_1 + k_2 + \cdots + k_t)),\]

where \(\beta = 1\) if \(e\) is even, while \(\beta = 0\) if \(e\) is odd.
Mass formula for self-orthogonal codes of the type \( \{k_1, k_2, \cdots, k_e\} \) and length \( n \) over \( \mathbb{R}_e \)

**Theorem [M. Yadav & A. ___ (2021)]**

Let \( N_e(n; k_1, k_2, \cdots, k_e) \) denote the number of distinct self-orthogonal codes of the type \( \{k_1, k_2, \cdots, k_e\} \) and length \( n \) over \( \mathbb{R}_e \). Let \( \mathbb{R}_e \cong \mathbb{F}_p^r \), where \( p \) is an odd prime and \( r \) is a positive integer.

- When \( e \) is odd, we have

\[
N_e(n; k_1, k_2, \cdots, k_{e-1}, k_e) = \begin{cases} 
\sigma_{p^r} \left( n, k_1 + k_2 + \cdots + k_{e+1} \right) \prod_{i=1}^{e+1} \left[ k_1 + k_2 + \cdots + k_i \right]_{p^r} \\
\times \prod_{j=2}^{e+1} \left[ k_j + k_{e+1} - k_1 \right]_{p^r} \sum_{\ell=1}^{e-1} n_{\ell}(k_1, k_2, \cdots, k_{e+1}) \\
&\text{if } k_1 \leq k_{e+1} \text{ and } k_s = k_{e-s+2} \text{ for } 2 \leq s \leq e; \\
0 & \text{otherwise.}
\end{cases}
\]
Mass formula for self-orthogonal codes of the type \( \{k_1, k_2, \cdots, k_e\} \) and length \( n \) over \( \mathcal{R}_e \) II

- When \( e \) is even, we have

\[
\mathcal{N}_e(n; k_1, k_2, \cdots, k_{e-1}, k_e)
\]

\[
= \begin{cases} 
\sigma_{pr} \left( n, k_1 + k_2 + \cdots + k_{\frac{e}{2}} \right) \prod_{i=1}^{\frac{e}{2}} \left[ k_i + \frac{k_{e+1}-k_1}{2} \right] p^r \\
\times \prod_{j=2}^{\frac{e}{2}+1} \left[ k_j + k_{e+1} - k_1 \right] \left( p^r \right)^{\ell=1} \sum_n n_{\ell}(k_1, k_2, \cdots, k_{\frac{e}{2}+1}) + \Theta_e^*(k_1, k_2, \cdots, k_{\frac{e}{2}+1}) \\
if \ k_1 \leq k_{e+1} \text{ and } k_s = k_{e-s+2} \text{ for } 2 \leq s \leq e; \\
0 \text{ otherwise,}
\end{cases}
\]

where

\[
\Theta_e^*(k_1, k_2, \cdots, k_{\frac{e}{2}+1}) = -(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \left( \frac{k_1 + k_2 + \cdots + k_{\frac{e}{2}} + 2k_{e+1} - 2k_1 - 1}{2} \right).
\]
Theorem [M. Yadav & A. (2021)]

Let \( \mathcal{N}_e(n) \) denote the number of distinct self-orthogonal codes of length \( n \) over \( \mathcal{R}_e \). Let \( \mathcal{R}_e \simeq \mathbb{F}_{p^r} \), where \( p \) is an odd prime and \( r \) is a positive integer.

- When \( e \) is odd, we have

\[
\mathcal{N}_e(n) = \sum_{k_1, k_2, \ldots, k_{\frac{e+1}{2}} \geq 0 \in \mathbb{N} \cup \{0\}} \sigma_{p^r} \left( n, k_1 + k_2 + \cdots + k_{\frac{e+1}{2}} \right)
\]

\[
0 \leq k_1 + k_2 + \cdots + k_{\frac{e+1}{2}} \leq \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
\times \prod_{i=1}^{\frac{e+1}{2}} \left[ k_1 + k_2 + \cdots + k_i \right]^{k_i} \left( p^r \right)^{\sum_{\ell=1}^{\frac{e-1}{2}} n_\ell(k_1, k_2, \ldots, k_{\frac{e+1}{2}})}
\]

\[
\times \prod_{j=2}^{\frac{e+1}{2}} \left[ k_j + n - 2(k_1 + k_2 + \cdots + k_{\frac{e+1}{2}}) \right]^{k_j} \left( p^r \right)^{\sum_{\ell=1}^{\frac{e-1}{2}} n_\ell(k_1, k_2, \ldots, k_{\frac{e+1}{2}})}.
\]
When $e$ is even, we have

$$N_e(n) = \sum_{k_1, k_2, \ldots, k_{\frac{e}{2}}+1 \in \mathbb{N} \cup \{0\}} \sigma_p r \left( n, k_1 + k_2 + \cdots + k_{\frac{e}{2}} \right)$$

$$\times \prod_{i=1}^{\frac{e}{2}} \left[ k_1 + k_2 + \cdots + k_i \right]_{k_i} \left( \sum_{\ell=1}^{\frac{e}{2}} n_\ell (k_1, k_2, \ldots, k_{\frac{e}{2}}+1) + \Theta_e(k_1, k_2, \ldots, k_{\frac{e}{2}}+1) \right)$$

$$\times \prod_{j=2}^{\frac{e}{2}+1} \left( k_j + n - 2(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) - k_{\frac{e}{2}}+1 \right)_{k_j} \left( \sigma_p r \right)_{k_j},$$

where

$$\Theta_e(k_1, k_2, \ldots, k_{\frac{e}{2}}+1) = -(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \left( \frac{2n - 3(k_1 + k_2 + \cdots + k_{\frac{e}{2}}) - 2k_{\frac{e}{2}}+1 - 1}{2} \right).$$

(Here $\mathbb{N}$ denotes the set of positive integers.)
Mass formula for self-dual codes of the type \( \{k_1, k_2, \ldots, k_e\} \) and length \( n \) over \( R_e \)

**Theorem [M. Yadav & A. ___ (2021)]**

Let \( M_e(n; k_1, k_2, \ldots, k_e) \) denote the number of distinct self-dual codes of the type \( \{k_1, k_2, \ldots, k_e\} \) and length \( n \) over \( R_e \). Let \( R_e \cong \mathbb{F}_{p^r} \), where \( p \) is an odd prime and \( r \) is a positive integer.

- When \( e \) is even, we have

\[
M_e(n; k_1, k_2, \ldots, k_e) = \begin{cases} 
\sigma_{p^r} \left( n, k_1 + k_2 + \cdots + k_{e/2} \right) \prod_{i=1}^{e/2} \left[ \frac{k_1 + k_2 + \cdots + k_i}{k_i} \right]^{p^r} \\
\sum_{\ell=1}^{e/2} h_\ell(k_1, k_2, \ldots, k_{e/2} + 1) + \lambda_e(k_1, k_2, \ldots, k_{e/2} + 1) \\
0 \quad \text{otherwise,}
\end{cases}
\]

where \( \lambda_e(k_1, k_2, \ldots, k_{e/2} + 1) = -(k_1 + k_2 + \cdots + k_{e/2}) \left( \frac{k_1 + k_2 + \cdots + k_{e/2}}{2} - 1 \right) \).
Mass formula for self-dual codes of the type \( \{k_1, k_2, \cdots, k_e\} \) and length \( n \) over \( \mathcal{R}_e \) II

- When \( e \) is odd, we have

\[
\mathcal{M}_e(n; k_1, k_2, \cdots, k_e) = \begin{cases} 
2 \prod_{b=1}^{\frac{n}{2}-1} (p^{rb} + 1) \prod_{i=1}^{\frac{e+1}{2}} \left[ \frac{k_1+k_2+\cdots+k_i}{k_i} \right] p^{r} \left( p^r \right)^{\frac{e-1}{2}} \sum_{\ell=1}^{\frac{e}{2}} h_{\ell}(k_1, k_2, \cdots, k_{\frac{e+1}{2}}) \\
\text{if } n \text{ is even, } (-1)^{\frac{n}{2}} \text{ is a square in } \mathcal{R}_e \text{ and } k_s = k_{e-s+2} \text{ for } 1 \leq s \leq e + 1; \\
0 \quad \text{otherwise.}
\end{cases}
\]
Theorem [M. Yadav & A. (2021)]

Let $M_e(n)$ denote the number of distinct self-dual codes of length $n$ over $\mathcal{R}_e$. Let $\mathcal{R}_e \simeq \mathbb{F}_{p^r}$, where $p$ is an odd prime and $r$ is a positive integer.

- When $e$ is even, we have

$$
M_e(n) = \sum_{k_1, k_2, \ldots, k_{\frac{e}{2}} \in \mathbb{N} \cup \{0\}} \sum_{0 \leq k_1 + k_2 + \cdots + k_{\frac{e}{2}} \leq \lfloor \frac{n}{2} \rfloor} \sigma_{p^r}(n, k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \prod_{i=1}^{\frac{e}{2}} \left[ \binom{k_1 + k_2 + \cdots + k_i}{k_i} \right]_{p^r} \left( \sum_{\ell=1}^{\frac{e}{2}-1} h_{\ell}(k_1, k_2, \ldots, k_{\frac{e}{2}}) + \lambda'_e(k_1, k_2, \ldots, k_{\frac{e}{2}}) \right)_{p^r},
$$

where $\lambda'_e(k_1, k_2, \ldots, k_{\frac{e}{2}}) = (k_1 + k_2 + \cdots + k_{\frac{e}{2}}) \left( \frac{k_1 + k_2 + \cdots + k_{\frac{e}{2}} - 1}{2} \right).$
When $e$ is odd, we have

$$
\mathcal{M}_e(n) = \begin{cases} 
\sum_{k_1, k_2, \ldots, k_{e-1} \in \mathbb{N} \cup \{0\}} & 2 \prod_{b=1}^{\frac{n}{2}-1} (p^{rb} + 1) \prod_{i=1}^{\frac{e-1}{2}} \left[ k_1 + k_2 + \cdots + k_i \right] p^r \\
0 \leq k_1 + k_2 + \cdots + k_{\frac{e-1}{2}} \leq \frac{n}{2} \\
\times \left[ k_1 + k_2 + \cdots + k_{\frac{e-1}{2}} \right] (p^r) & \frac{e-3}{2} \sum_{\ell=1}^{\frac{e-1}{2}} h_\ell(k_1, k_2, \ldots, k_{\frac{e-1}{2}}) + \lambda^*_e(k_1, k_2, \ldots, k_{\frac{e-1}{2}}) \\
\text{if } n \text{ is even and } (-1)^{\frac{n}{2}} \text{ is a square in } \overline{\mathbb{R}}_e; \\
0 & \text{otherwise,}
\end{cases}
$$

where $\lambda_e^*(k_1, k_2, \ldots, k_{\frac{e-1}{2}}) = \left( \frac{n}{2} - 1 \right) \left( k_1 + k_2 + \cdots + k_{\frac{e-1}{2}} \right)$.
Theorem [Bhowmick et al. (2020)]

Any LCD code $C$ of length $n$ over the finite commutative Frobenius ring $R$ is a free code, i.e., the code $C$ is a free $R$-submodule of $R^n$.

Any LCD code $C$ of length $n$ over the finite commutative chain ring $R_e$ is a free code, i.e., the code $C$ is a free $R_e$-submodule of $R_e^n$.

As a consequence, the LCD code $C$ is permutation equivalent to a code whose generator matrix $G$ is in the standard form

$$G = [I_k \mid A],$$

where $I_k$ is the $k \times k$ identity matrix and $A$ is a $k \times (n - k)$ matrix over $R_e$.

The integer $k$ is called the rank of the code $C$. 
Theorem [M. Yadav & A. _____ (2021)]

For $0 \leq k \leq n$, let $L_e(n; k)$ denote the number of distinct LCD codes of length $n$ and rank $k$ over $R_e$. Let $R_e \cong F_{p^r}$, where $p$ is a prime and $r$ is a positive integer. Then we have $L_e(n; 0) = L_e(n; n) = 1$. Further, for $1 \leq k \leq n - 1$, we have the following:

- When $p = 2$, we have

$$L_e(n; k) = \left\{ \begin{array}{ll}
2 \frac{r(n-k)(2k\ell-k+1)}{2} \left[ \frac{(n-1)/2}{(k-1)/2} \right]_{2^2r} & \text{if both } k \text{ and } n \text{ are odd;} \\
2 \frac{r(k(n-k)(2\ell-1)+n-1)}{2} \left[ \frac{(n-2)/2}{(k-1)/2} \right]_{2^2r} & \text{if } k \text{ is odd and } n \text{ is even;} \\
2 \frac{r(k(n-k)(2\ell-1)+1)}{2} \left[ \frac{(n-1)/2}{k/2} \right]_{2^2r} & \text{if } k \text{ is even and } n \text{ is odd;} \\
2 \frac{r(k(n-k)(2\ell-1)-2)}{2} \left(2^{rk} + 2^r - 1 \right) \left[ \frac{(n-2)/2}{k/2} \right]_{2^2r} \\
+ (2^r(n-k+1) - 2^r(n-k) + 1) \left[ \frac{(n-2)/2}{(k-2)/2} \right]_{2^2r} & \text{if both } k \text{ and } n \text{ are even.}
\end{array} \right.$$
When $p$ is an odd prime, we have

$$
\mathcal{L}_e(n; k) = \begin{cases} 
p^\frac{r(n-k)(2k\ell-k+1)}{2} \left[\frac{(n-1)/2}{(k-1)/2}\right] p^{2r} & \text{if both } k \text{ and } n \text{ are odd;} 
p^\frac{r(k(n-k)(2\ell-1)-1)}{2} \left(p^\frac{rn}{2} - 1\right) \left[\frac{(n-2)/2}{(k-1)/2}\right] p^{2r} & \text{if } k \text{ is odd and } n \text{ is even} 
\end{cases}
$$

with either $p^r \equiv 1 \pmod{4}$ or $n \equiv 0 \pmod{4}$ and $p^r \equiv 3 \pmod{4}$;

$$
p^\frac{r(k(n-k)(2\ell-1)-1)}{2} \left(p^\frac{rn}{2} + 1\right) \left[\frac{(n-2)/2}{(k-1)/2}\right] p^{2r} & \text{if } k \text{ is odd, } n \text{ is even,} 
p^r \equiv 3 \pmod{4} \text{ and } n \equiv 2 \pmod{4};
\end{cases}
$$

$$
p^\frac{r(k(n-k)(2\ell-1)+1)}{2} \left[\frac{(n-1)/2}{k/2}\right] p^{2r} & \text{if } k \text{ is even and } n \text{ is odd;}
\end{cases}
$$

$$
p^\frac{r(k(n-k)(2\ell-1))}{2} \left[\frac{n/2}{k/2}\right] p^{2r} & \text{if both } k \text{ and } n \text{ are even.}
\end{cases}
$$
Mass formula for LCD codes of length $n$ over $\mathbb{R}_e$

**Theorem [M. Yadav & A. (2021)]**

Let $L_e(n)$ denote the number of distinct LCD codes of length $n$ over $\mathbb{R}_e$. Let $\mathbb{R}_e \cong \mathbb{F}_{p^r}$, where $p$ is a prime and $r$ is a positive integer.

- When $p = 2$, we have

$$L_e(n) = \begin{cases} 
2 + \sum_{k=1}^{n-1} 2^{r} \frac{k((n-k)(2\ell-1)+1)}{2} \left[ \frac{(n-1)/2}{k/2} \right]_{22r} & \text{if } n \text{ is odd;} \\
2 + \sum_{k=1}^{n-1} 2^{r} \frac{k((n-k)(2\ell-1)+1)}{2} \left[ \frac{(n-1)/2}{(k-1)/2} \right]_{22r} & \text{if } n \text{ is even.}
\end{cases}$$
When $p$ is an odd prime and $n$ is even, we have

$$\mathcal{L}_e(n) = \begin{cases} 
2 + \sum_{k=1}^{n-1} \frac{r k (n-k) (2\ell - 1)}{2} \left[ \frac{n}{2} \right] \frac{[k/2] p^{2r}}{p^{2r}} & \text{if either } p^r \equiv 1 \pmod{4} \text{ or } n \equiv 0 \pmod{4} \text{ and } p^r \equiv 3 \pmod{4}; \\
+ \sum_{k=1}^{n-1} \frac{r (k(n-k)(2\ell - 1) - 1)}{2} \left( p^{\frac{rn}{2}} - 1 \right) \left[ \frac{(n-2)/2}{(k-1)/2} \right] p^{2r} & \text{if } p^r \equiv 3 \pmod{4} \text{ and } n \equiv 2 \pmod{4}\end{cases}$$
When $p$ is an odd prime and $n$ is odd, we have

\[
\mathcal{L}_e(n) = 2 + \sum_{k=1 \atop k \equiv 1 \pmod{2}}^{n-1} \frac{r(n-k)(2k\ell-k+1)}{2} \cdot p^{(n-1)/2} \left\lfloor \frac{(n-1)/2}{(k-1)/2} \right\rfloor_{p^{2r}}
\]

\[
+ \sum_{k=1 \atop k \equiv 0 \pmod{2}}^{n-1} \frac{rk((n-k)(2\ell-1)+1)}{2} \cdot p^{(n-1)/2} \left\lfloor \frac{k/2}{k/2} \right\rfloor_{p^{2r}}.
\]
Two self-orthogonal (resp. self-dual, LCD) codes of length $n$ over $\mathcal{R}_e$ are said to be equivalent if one code can be obtained from the other by a combination of operations of the following two types:

A. Permutation of the $n$ coordinate positions of the code.

B. Multiplication of the code symbols appearing in a given coordinate position by the element $-1 \in \mathcal{R}_e$. 
Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 3 over $\mathbb{F}_5[u]/<u^2>$

<table>
<thead>
<tr>
<th>Type ${k_1, k_2}$</th>
<th>Total number of self-orthogonal codes of the type ${k_1, k_2}$ and length 3 over $\mathbb{F}_5[u]/&lt;u^2&gt;$</th>
<th>Number of inequivalent self-orthogonal codes of the type ${k_1, k_2}$ and length 3 over $\mathbb{F}_5[u]/&lt;u^2&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 0}$</td>
<td>30</td>
<td>2</td>
</tr>
<tr>
<td>${1, 1}$</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>${0, 1}$</td>
<td>31</td>
<td>5</td>
</tr>
<tr>
<td>${0, 2}$</td>
<td>31</td>
<td>5</td>
</tr>
<tr>
<td>${0, 3}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

There are precisely 14 inequivalent non-zero self-orthogonal codes of length 3 over $\mathbb{F}_5[u]/<u^2>$ with generator matrices:

- $uI_3$, $[1 \quad 0 \quad 2]$, $[1 \quad u \quad 2]$, $[u \quad 0 \quad 0]$, $[u \quad u \quad u]$, $[u \quad 2u \quad u]$, $[u \quad u \quad 0]$, $[u \quad 0 \quad 3u]$, $[1 \quad 0 \quad 2]$, $[u \quad 0 \quad 0]$, $[0 \quad u \quad 0]$, $[u \quad 0 \quad 4u]$, $[u \quad 0 \quad 4u]$. 
Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 4 over $\mathbb{F}_5[u]/ < u^2 >$

<table>
<thead>
<tr>
<th>Type ${k_1, k_2}$</th>
<th>Total number of self-orthogonal codes of the type ${k_1, k_2}$ and length 4 over $\mathbb{F}_5[u]/ &lt; u^2 &gt;$</th>
<th>Number of inequivalent self-orthogonal codes of the type ${k_1, k_2}$ and length 4 over $\mathbb{F}_5[u]/ &lt; u^2 &gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 0}$</td>
<td>900</td>
<td>10</td>
</tr>
<tr>
<td>${1, 1}$</td>
<td>1080</td>
<td>14</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>36</td>
<td>2</td>
</tr>
<tr>
<td>${2, 0}$</td>
<td>60</td>
<td>2</td>
</tr>
<tr>
<td>${0, 1}$</td>
<td>156</td>
<td>8</td>
</tr>
<tr>
<td>${0, 2}$</td>
<td>806</td>
<td>18</td>
</tr>
<tr>
<td>${0, 3}$</td>
<td>156</td>
<td>8</td>
</tr>
<tr>
<td>${0, 4}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
There are precisely 63 inequivalent non-zero self-orthogonal codes of length 4 over \( \mathbb{F}_5[u]/<u^2> \), whose generator matrices are as listed below:

- \[
\begin{bmatrix}
1 & xu & yu & 2 \\
0 & u & yu & 0 \\
\end{bmatrix}
\text{ with } (x, y) \in \{(0, 0), (1, 0), (1, 1), (1, 2), (2, 0)\};
\]

- \[
\begin{bmatrix}
1 & xu + 1 & yu + 2 & zu + 2 \\
0 & u & zu & wu \\
\end{bmatrix}
\text{ with } (x, y, z) \in \{(0, 0, 0), (0, 1, 4), (1, 0, 2), (1, 3, 4), (0, 3, 2), (2, 4, 0)\};
\]

- \[
\begin{bmatrix}
1 & 0 & xu & 2 \\
0 & u & xu & 0 \\
\end{bmatrix}
\text{ with } (x, y) \in \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2)\};
\]

- \[
\begin{bmatrix}
1 & 1 & xu + 2 & yu + 2 \\
0 & u & zu & wu \\
\end{bmatrix}
\text{ with } (x, y, z, w) \in \{(0, 0, 0, 2), (1, 4, 0, 2), (2, 3, 0, 2), (0, 0, 1, 1), (1, 4, 1, 1), (3, 2, 1, 1), (0, 0, 3, 4), (1, 4, 3, 4)\};
\]

- \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 2 & 2 \\
0 & 1 & 2 & 4u \\
\end{bmatrix}
\]

- \[
\begin{bmatrix}
1 & u & 0 & 2 \\
0 & u & 0 & 2 \\
0 & 0 & u & 0 \\
0 & 0 & u & 4u \\
\end{bmatrix}
\]

- \[
\begin{bmatrix}
1 & 0 & xu & yu & zu \\
0 & u & xu & yu & zu \\
\end{bmatrix}
\text{ with } (x, y, z) \in \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 3, 4), (1, 2, 4), (2, 1, 0), (2, 0, 0)\};
\]

- \[
\begin{bmatrix}
u & 0 & xu & yu \\
0 & u & zu & wu \\
\end{bmatrix}
\text{ with } (x, y, z, w) \in \{(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (3, 0, 2, 0), (0, 3, 1, 2), (1, 0, 0, 1), (4, 0, 2, 1), (3, 4, 2, 1), (1, 0, 0, 2), (0, 0, 0, 2), (0, 0, 1, 2), (4, 2, 1, 2), (4, 2, 2, 2), (4, 2, 2, 1), (4, 4, 4, 0), (4, 4, 4, 1), (2, 0, 0, 2), (1, 0, 1, 0)\};
\]

- \[
\begin{bmatrix}
u & 0 & 0 & xu \\
0 & u & 0 & yu \\
0 & 0 & u & zu \\
\end{bmatrix}
\text{ with } (x, y, z) \in \{(0, 0, 0), (1, 0, 0), (4, 1, 0), (2, 0, 0), (3, 1, 0), (1, 4, 1), (4, 4, 3), (2, 4, 3)\}.\]
Total number of self-orthogonal and inequivalent self-orthogonal codes of a given type and length 5 over $\mathbb{F}_5[u]/<u^2>$

<table>
<thead>
<tr>
<th>Type ${k_1, k_2}$</th>
<th>Total number of self-orthogonal codes of the type ${k_1, k_2}$ and length 5 over $\mathbb{F}_5[u]/&lt;u^2&gt;$</th>
<th>Number of inequivalent self-orthogonal codes of the type ${k_1, k_2}$ and length 5 over $\mathbb{F}_5[u]/&lt;u^2&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 0}$</td>
<td>19500</td>
<td>27</td>
</tr>
<tr>
<td>${1, 1}$</td>
<td>120900</td>
<td>109</td>
</tr>
<tr>
<td>${1, 2}$</td>
<td>24180</td>
<td>41</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>156</td>
<td>3</td>
</tr>
<tr>
<td>${0, 1}$</td>
<td>781</td>
<td>11</td>
</tr>
<tr>
<td>${0, 2}$</td>
<td>20306</td>
<td>49</td>
</tr>
<tr>
<td>${0, 3}$</td>
<td>20306</td>
<td>49</td>
</tr>
<tr>
<td>${0, 4}$</td>
<td>781</td>
<td>11</td>
</tr>
<tr>
<td>${0, 5}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>${2, 0}$</td>
<td>19500</td>
<td>16</td>
</tr>
<tr>
<td>${2, 1}$</td>
<td>780</td>
<td>4</td>
</tr>
</tbody>
</table>
There are precisely 321 inequivalent non-zero self-orthogonal codes of length 5 over \( \mathbb{F}_5[u]/<u^2> \), whose generator matrices are as listed below:

- \[
\begin{bmatrix}
1 & xu & yu & zu & 2
\end{bmatrix}
\text{ with } (x, y, z) \in \{ (0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 2), (1, 1, 1), (1, 1, 2) \};
\]

- \[
\begin{bmatrix}
1 & xu & yu + 1 & zu + 2 & wu + 2
\end{bmatrix}
\text{ with } (x, y, z, w) \in \{ (0, 0, 0, 0), (0, 0, 1, 4), (0, 2, 3), (0, 1, 0, 2), (0, 1, 3, 4), (0, 2, 0, 4), (3, 0, 2, 3), (1, 0, 0, 0), (1, 0, 1, 4), (1, 0, 2, 3), (1, 1, 0, 2), (1, 1, 1, 1), (1, 1, 3, 4), (1, 2, 0, 4), (1, 2, 1, 3) \};
\]

- \[
\begin{bmatrix}
1 & xu + 1 & yu + 1 & zu + 1 & wu + 1
\end{bmatrix}
\text{ with } (x, y, z, w) \in \{ (0, 0, 0, 0), (0, 0, 1, 4), (0, 0, 2, 3), (0, 1, 1, 3), (0, 1, 2, 2), (1, 2, 3, 4) \};
\]

- \[
\begin{bmatrix}
1 & 0 & xu & yu & 2
0 & u & zu & wu & 0
\end{bmatrix}
\text{ with } (x, y, z, w) \in \{ (0, 0, 0, 0), (0, 1, 0, 0), (1, 1, 0, 0), (1, 2, 0, 0), (4, 2, 0, 1), (0, 0, 1, 0), (0, 1, 0, 1), (1, 0, 0, 1), (1, 1, 0, 1), (4, 2, 1, 1), (0, 0, 1, 1), (0, 1, 1, 1), (0, 0, 2, 0), (0, 1, 1, 2), (0, 1, 2, 2), (0, 1, 2, 0), (0, 1, 2, 1), (1, 1, 4, 3), (1, 1, 0, 2), (1, 1, 1, 3) \};
\]

- \[
\begin{bmatrix}
1 & 0 & 1 & 2 & 2
0 & u & xu & yu & zu
\end{bmatrix}
\text{ with } (x, y, z) \in \{ (0, 0, 0), (0, 1, 4), (0, 2, 3), (1, 0, 2), (1, 3, 4), (2, 1, 3) \};
\]

- \[
\begin{bmatrix}
1 & 0 & 1 & u + 2 & 4u + 2
0 & u & xu & yu & zu
\end{bmatrix}
\text{ with } (x, y, z) \in \{ (0, 0, 0), (0, 1, 4), (0, 2, 3), (1, 0, 2), (1, 1, 1), (1, 3, 4), (2, 0, 4), (2, 1, 3), (3, 3, 3) \};
\]

- \[
\begin{bmatrix}
1 & 0 & 1 & 2u + 2 & 3u + 2
0 & u & xu & yu & zu
\end{bmatrix}
\text{ with } (x, y, z) \in \{ (0, 0, 0), (0, 1, 4), (0, 2, 3), (1, 0, 2), (1, 1, 1), (1, 3, 4), (2, 0, 4), (2, 1, 3), (2, 2, 2) \};
\]
Some preliminaries
Mass formulae
Classification

Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
1 & 0 & u + 1 & 2 & 2u + 2 \\
0 & u & xu & yu & zu
\end{bmatrix}
\]
\[
\text{with } (x, y, z) \in \{(0, 0, 0), (0, 1, 4), (0, 2, 3), (1, 0, 2),
(1, 2, 0), (1, 3, 4), (2, 1, 3)\};
\]

\[
\begin{bmatrix}
1 & 0 & u + 1 & 3u + 2 & 4u + 2 \\
0 & u & xu & yu & zu
\end{bmatrix}
\]
\[
\text{with } (x, y, z) \in \{(0, 0, 0), (0, 1, 4), (0, 2, 3),
(1, 0, 2), (1, 1, 1), (1, 2, 0), (1, 4, 3), (1, 3, 4), (2, 0, 4), (2, 1, 3), (2, 2, 2), (2, 3, 1), (2, 4, 0)\};
\]

\[
\begin{bmatrix}
1 & 0 & 2u + 1 & 2 & 4u + 2 \\
0 & u & xu & yu & zu
\end{bmatrix}
\]
\[
\text{with } (x, y, z) \in \{(0, 0, 0), (0, 1, 4), (0, 2, 3), (1, 0, 2),
(1, 2, 0), (1, 3, 4), (2, 1, 3)\};
\]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & u & xu & yu & zu
\end{bmatrix}
\]
\[
\text{with } (x, y, z) \in \{(0, 0, 4), (0, 1, 3), (2, 3, 4)\};
\]

\[
\begin{bmatrix}
1 & 1 & 1 & u + 1 & 4u + 1 \\
0 & u & xu & yu & zu
\end{bmatrix}
\]
\[
\text{with } (x, y, z) \in \{(0, 0, 4), (0, 1, 3), (0, 2, 2), (0, 4, 0),
(2, 0, 2), (2, 1, 1), (2, 3, 4)\};
\]

\[
\begin{bmatrix}
1 & 1 & xu + 1 & yu + 1 & 3u + 1 \\
0 & u & zu & wu & su
\end{bmatrix}
\]
\[
\text{with } (x, y, z, w, s) \in \{(0, 2, 0, 0, 4),
(0, 2, 0, 4, 0), (0, 2, 2, 0, 2), (0, 2, 2, 1, 1), (1, 1, 0, 2, 2), (1, 1, 2, 4, 3)\};
\]

\[
\begin{bmatrix}
1 & 3 & 1 & 3 & xu \\
0 & u & yu & zu & wu
\end{bmatrix}
\]
\[
\text{with } (x, y, z, w) \in \{(1, 0, 4, 0), (1, 1, 2, 0), (1, 2, 0, 0),
(1, 4, 1, 0), (0, 0, 4, 0), (0, 1, 2, 0), (0, 2, 0, 0)\};
\]
Some preliminaries

Self-orthogonal codes

Mass formulae

Self-dual codes

Classification

LCD codes

\[
\begin{bmatrix}
1 & 3 & u + 1 & 3u + 3 & xu \\
0 & u & yu & zu & wu
\end{bmatrix}
\]

with \((x, y, z, w) \in \{(0, 0, 4, 0), (0, 1, 2, 0), (0, 2, 0, 0), (1, 0, 4, 0), (1, 1, 2, 0), (1, 3, 3, 0), (1, 4, 1, 0), (1, 2, 0, 0)\}\);

\[
\begin{bmatrix}
1 & 3 & 3u + 1 & 4u + 3 & xu \\
0 & u & yu & zu & wu
\end{bmatrix}
\]

with \((x, y, z, w) \in \{(0, 2, 0, 0), (1, 2, 0, 0), (1, 0, 4, 0)\}\);

\[uI_5, \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & u + 2 & 3u + 3 & u + 4
\end{bmatrix}, \begin{bmatrix}
1 & 1 & u + 1 & 2u + 1 & 2u + 1 \\
0 & u & 3u & 2u & 4u
\end{bmatrix}\];

\[
\begin{bmatrix}
1 & 3 & 2u + 1 & u + 3 & xu \\
0 & u & 4u & u & 0
\end{bmatrix}
\]

with \(x \in \{0, 1\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & u & 0 & 0 & 0 \\
0 & 0 & u & 0 & 0 \\
0 & 0 & 0 & u & 0 \\
0 & 0 & 0 & u & 2u
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 1 & 2 & 2 \\
0 & u & 0 & 0 & 0 \\
0 & u & 0 & 0 & u \\
0 & 0 & 0 & u & 4u
\end{bmatrix}, \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & u & 0 & 0 & 0 \\
0 & 0 & u & u & 3u \\
0 & 0 & 0 & u & 4u
\end{bmatrix}\];

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 2 \\
0 & u & 0 & yu & 0 \\
0 & 0 & u & zu & 0
\end{bmatrix}
\]

with \((x, y, z) \in \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (1, 0, 0), (1, 0, 1), (1, 0, 2), (3, 1, 1), (4, 1, 2), (0, 1, 3), (0, 1, 1)\}\);
Some preliminaries

Mass formulae

Classification

Self-orthogonal codes

Self-dual codes

LCD codes

\[
\begin{bmatrix}
1 & 0 & 1 & xu + 2 & yu + 2 \\
0 & u & 0 & zu & wu \\
0 & 0 & u & vu & su \\
\end{bmatrix}
\]

with \((x, y, z, w, v, s) \in \{(0, 0, 0, 0, 2), (0, 0, 0, 1, 1), (0, 0, 0, 3, 4), (0, 0, 1, 4, 3, 4), (0, 0, 1, 4, 0, 2), (0, 0, 1, 4, 1, 1), (1, 4, 1, 4, 0, 2), (0, 0, 2, 3, 3, 4), (1, 4, 0, 0, 2, 0), (1, 4, 0, 0, 1, 1), (1, 4, 0, 0, 3, 4), (1, 4, 1, 4, 4, 3), (1, 4, 1, 4, 1, 1), (1, 4, 2, 3, 3, 4), (1, 4, 2, 3, 1, 1), (2, 3, 0, 0, 2, 0), (2, 3, 0, 0, 1, 1), (2, 3, 1, 4, 1, 1), (2, 3, 1, 4, 0, 2), (2, 3, 2, 3, 1, 1)\};

\[
\begin{bmatrix}
1 & 1 & 1 & xu + 1 & yu + 1 \\
0 & u & 0 & zu & wu \\
0 & 0 & u & vu & su \\
\end{bmatrix}
\]

with \((x, y, z, w, v, s) \in \{(0, 0, 0, 4, 1, 3), (0, 0, 0, 4, 2, 2), (1, 4, 0, 4, 4, 0), (1, 4, 0, 4, 0, 4), (0, 0, 1, 3, 3, 1), (2, 3, 0, 4, 1, 3), (2, 3, 0, 4, 2, 2), (1, 4, 1, 3, 3, 1)\};

\[
\begin{bmatrix}
1 & 3 & 1 & 3 & xu \\
0 & u & 0 & 4u & 0 \\
0 & 0 & u & 3u & 0 \\
\end{bmatrix}
\]

with \(x \in \{0, 1\};

[\[
\begin{bmatrix}
u & xu & yu & zu & wu \\
\end{bmatrix}
\]

with \((x, y, z, w) \in \{(0, 0, 0, 2), (0, 1, 2, 2), (1, 1, 1, 1), (0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (2, 1, 1, 0), (2, 3, 1, 1), (2, 3, 3, 3), (2, 1, 0, 0)\};
\[
\begin{bmatrix}
u & 0 & xu & yu & zu \\
0 & u & wu & vu & su
\end{bmatrix}
\text{with } (x, y, z, w, v, s) \in \{(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 2), (0, 0, 0, 0, 1, 1), (0, 0, 0, 0, 1, 2), (0, 0, 0, 1, 1, 1), (0, 0, 0, 1, 1, 2), (0, 0, 0, 1, 3, 3), (0, 0, 1, 4, 3, 1), (0, 0, 1, 0, 0, 1), (0, 0, 1, 0, 0, 2), (0, 0, 1, 0, 1, 0), (0, 0, 1, 0, 1, 2), (0, 0, 1, 0, 2, 0), (0, 0, 1, 0, 2, 1), (0, 0, 1, 1, 1, 2), (0, 0, 1, 1, 2, 0), (0, 0, 1, 1, 2, 2), (0, 0, 1, 1, 4, 0), (0, 1, 1, 3, 4, 0), (0, 1, 1, 3, 4, 1), (0, 1, 1, 4, 2, 1), (0, 1, 1, 1, 1, 3), (0, 1, 1, 1, 2, 0), (0, 1, 1, 1, 2, 2), (0, 1, 1, 2, 3, 3), (0, 1, 2, 2, 4, 3), (0, 1, 2, 2, 4, 3), (0, 1, 2, 0, 2, 0), (0, 1, 2, 1, 0, 2), (0, 1, 2, 1, 1, 1), (0, 1, 2, 1, 1, 2), (0, 1, 2, 1, 2, 0), (0, 1, 2, 1, 2, 1), (0, 1, 2, 1, 2, 2), (0, 1, 2, 2, 0, 0), (0, 1, 2, 2, 1, 1), (0, 1, 2, 2, 1, 3), (2, 3, 2, 0, 0, 2), (1, 2, 3, 1, 1, 1), (1, 2, 2, 1, 3, 4), (1, 2, 2, 2, 1, 3), (1, 1, 1, 0, 0, 2), (0, 0, 2, 3, 0, 0)\};
\]

\[
\begin{bmatrix}
u & 0 & 0 & xu & yu \\
0 & u & 0 & zu & wz \\
0 & 0 & u & vu & su
\end{bmatrix}
\text{with } (x, y, z, w, v, s) \in \{(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 2), (0, 0, 0, 0, 1, 1), (0, 0, 0, 0, 1, 2), (0, 0, 0, 1, 4, 4), (0, 0, 0, 1, 1), (0, 0, 0, 1, 0, 2), (0, 0, 0, 1, 1, 0), (0, 0, 0, 1, 1, 2), (0, 0, 0, 1, 2, 0), (0, 0, 0, 2, 1, 4), (0, 0, 0, 2, 1, 4), (0, 0, 0, 2, 2, 3), (0, 0, 0, 2, 2, 0), (0, 0, 0, 1, 1, 2), (0, 0, 0, 1, 1, 3), (0, 0, 0, 1, 1, 3), (0, 0, 2, 3, 4, 3), (0, 1, 2, 0, 4, 4), (0, 1, 3, 2, 0, 1), (0, 1, 3, 3, 0, 2), (0, 1, 2, 0, 4, 4), (0, 1, 4, 2, 1, 1), (0, 1, 4, 3, 1, 2), (0, 1, 4, 3, 3), (0, 1, 3, 3, 3, 2), (0, 1, 1, 2, 3, 1), (0, 1, 0, 1, 4, 4), (0, 1, 0, 1, 0, 1), (0, 1, 0, 1, 0, 2), (0, 1, 0, 1, 1, 3), (0, 1, 0, 1, 1, 3), (0, 1, 0, 2, 4, 3), (0, 1, 0, 2, 0, 2), (0, 1, 0, 2, 1, 4), (0, 1, 0, 2, 1, 4), (0, 1, 0, 2, 0, 2), (0, 1, 0, 2, 4, 3), (0, 1, 0, 1, 4, 2, 4), (0, 1, 1, 4, 2, 4), (0, 1, 1, 0, 2, 2), (1, 1, 4, 3, 4, 1), (1, 1, 4, 3, 0, 2), (1, 1, 4, 3, 2, 4), (2, 2, 3, 0, 4, 4), (2, 2, 1, 4, 0, 2), (2, 2, 4, 2, 1, 3), (1, 2, 2, 1, 1, 4), (1, 2, 2, 1, 2, 3)\};
\]
Some preliminaries
Mass formulae
Classification
Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
u & 0 & 0 & 0 & xu \\
0 & u & 0 & 0 & yu \\
0 & 0 & u & 0 & zu \\
0 & 0 & 0 & u & wu \\
\end{bmatrix}
\text{with } (x, y, z, w) \in \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 0, 2), (0, 0, 1, 4), (0, 0, 1, 2), (0, 1, 1, 2), (0, 1, 2, 3), (0, 1, 1, 4), (1, 1, 1, 2), (1, 4, 2, 3), (1, 4, 1, 1)\};
\]

\[
\begin{bmatrix}
1 & 0 & xu & yu & 2 \\
0 & 1 & zu & 2 & wu \\
\end{bmatrix}
\text{with } (x, y, z, w) \in \{(0, 0, 0, 0), (0, 0, 1, 0), (0, 1, 0, 4), (0, 1, 1, 4), (1, 0, 1, 0), (1, 1, 1, 4), (1, 1, 2, 4)\};
\]

\[
\begin{bmatrix}
1 & 0 & 1 & xu + 2 & yu + 2 \\
0 & 1 & zu + 2 & wu + 1 & su + 3 \\
\end{bmatrix}
\text{with } (x, y, z, w, s) \in \{(0, 0, 0, 0, 0), (0, 0, 1, 4, 3), (0, 0, 2, 3, 1), (1, 4, 0, 4, 2), (1, 4, 1, 3, 0), (1, 4, 2, 2, 3), (2, 3, 0, 3, 4), (2, 3, 2, 1, 0)\};
\]

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 2 \\
0 & 1 & 0 & 2 & yu \\
0 & 0 & u & 0 & 0 \\
\end{bmatrix}
\text{with } (x, y) \in \{(0, 0), (1, 4)\};
\]

\[
\begin{bmatrix}
1 & 0 & 1 & xu + 2 & yu + 2 \\
0 & 1 & 2 & zu + 1 & wu + 3 \\
0 & 0 & u & 4u & 3u \\
\end{bmatrix}
\text{with } (x, y, z, w) \in \{(0, 0, 0, 0), (1, 4, 4, 2)\}.
A self-orthogonal code of the type \( \{k_1, k_2\} \) and length \( n \) over \( \mathbb{F}_q[u]/<u^2> \) is self-dual if and only if

\[
2k_1 + k_2 = n.
\]

There are precisely

- 2 inequivalent self-dual codes of length 3 over \( \mathbb{F}_5[u]/<u^2> \).
- 5 inequivalent self-dual codes of length 4 over \( \mathbb{F}_5[u]/<u^2> \).
- 8 inequivalent self-dual codes of length 5 over \( \mathbb{F}_5[u]/<u^2> \).
Total number of LCD and inequivalent LCD codes of a given rank and length 4 over $\mathbb{F}_2[u]/\langle u^2 \rangle$

<table>
<thead>
<tr>
<th>Rank $k$</th>
<th>Total number of LCD codes of length 4 and rank $k$ over $\mathbb{F}_2[u]/\langle u^2 \rangle$</th>
<th>Number of inequivalent LCD codes of length 4 and rank $k$ over $\mathbb{F}_2[u]/\langle u^2 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>64</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>320</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>64</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
There are precisely 41 inequivalent non-zero LCD codes of length 4 over \( \mathbb{F}_2[u]/<u^2> \), whose generator matrices are as listed below:

- \[
\begin{bmatrix}
1 & xu & yu & zu
\end{bmatrix}
\] with \((x, y, z) \in \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)\};

- \[
\begin{bmatrix}
1 & 1+xu & 1+yu & zu
\end{bmatrix}
\] with \((x, y, z) \in \{(1, 1, 1), (1, 1, 0), (0, 0, 1), (0, 0, 0)\};

- \[
\begin{bmatrix}
1 & 0 & xu & yu \\
0 & 1 & zu & wu
\end{bmatrix}
\] with \((x, y, z, w) \in \{(0, 0, 0, 0), (1, 0, 0, 1), (1, 1, 0, 1), (1, 1, 1, 1), (0, 1, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1)\};

- \[
\begin{bmatrix}
1 & 0 & 1+xu & 1+yu \\
0 & 1 & zu & wu
\end{bmatrix}
\] with \((x, y, z, w) \in \{(0, 0, 0, 0), (1, 0, 1, 1), (1, 1, 1, 1), (0, 1, 0, 0)\};

- \[
\begin{bmatrix}
1 & 0 & 1+xu & 1+yu \\
0 & 1 & zu & 1+wu
\end{bmatrix}
\] with \((x, y, z, w) \in \{(0, 0, 0, 0), (1, 1, 1, 1)\};

- \[
\begin{bmatrix}
1 & 0 & 1+xu & 1+yu \\
0 & 1 & 1+zu & 1+wu
\end{bmatrix}
\] with \((x, y, z, w) \in \{(1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\};

- \[
\begin{bmatrix}
1 & 0 & 1+xu & yu \\
0 & 1 & 1+zu & wu
\end{bmatrix}
\] with \((x, y, z, w) \in \{(0, 1, 0, 0), (0, 1, 1, 1), (0, 0, 0, 0), (0, 0, 1, 0), (0, 0, 1, 1)\};

- \[
\begin{bmatrix}
I_4 \\
1 & 0 & u & u \\
0 & 1 & 1 & 1
\end{bmatrix}
\]
Some preliminaries

Mass formulae

Classification

Self-orthogonal codes

Self-dual codes

LCD codes

\begin{align*}
\begin{bmatrix}
1 & 0 & 0 & xu \\
0 & 1 & 0 & yu \\
0 & 0 & 1 & zu \\
\end{bmatrix} & \text{ with } (x, y, z) \in \{(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1)\}; \\
\begin{bmatrix}
1 & 0 & 0 & 1 + xu \\
0 & 1 & 0 & 1 + yu \\
0 & 0 & 1 & zu \\
\end{bmatrix} & \text{ with } (x, y, z) \in \{(1, 1, 1), (1, 1, 0), (0, 0, 0), (0, 0, 1)\}.
\end{align*}
Total number of LCD and inequivalent LCD codes of a given rank and length 5 over $\mathbb{F}_2[u]/<u^2>$

<table>
<thead>
<tr>
<th>Rank $k$</th>
<th>Total number of LCD codes of length 5 and rank $k$ over $\mathbb{F}_2[u]/&lt;u^2&gt;$</th>
<th>Number of inequivalent LCD codes of length 5 and rank $k$ over $\mathbb{F}_2[u]/&lt;u^2&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>256</td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td>5120</td>
<td>81</td>
</tr>
<tr>
<td>3</td>
<td>5120</td>
<td>81</td>
</tr>
<tr>
<td>4</td>
<td>256</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
There are precisely 191 inequivalent non-zero LCD codes of length 5 over \( \mathbb{F}_2[u]/<u^2> \), whose generator matrices are as listed below:

- \[
\begin{bmatrix}
1 & xu & yu & zu & wu
\end{bmatrix}
\]  with \( (x, y, z, w) \in \{(0, 1, 0, 1), (0, 0, 1, 0), (1, 0, 1, 1), (0, 0, 0, 0), (1, 1, 1, 1)\};

- \[
\begin{bmatrix}
1 & 1 + xu & 1 + yu & zu & wu
\end{bmatrix}
\]  with \( (x, y, z, w) \in \{(0, 0, 0, 0), (1, 0, 1, 1), (0, 0, 0, 1), (1, 0, 1, 1), (1, 1, 0, 0), (0, 0, 1, 1)\};

- \[
\begin{bmatrix}
1 & 1 + xu & 1 + yu & 1 + zu & 1 + wu
\end{bmatrix}
\]  with \( (x, y, z, w) \in \{(0, 0, 1, 1), (1, 0, 0, 0), (0, 0, 0, 0)\};

- \[
\begin{bmatrix}
1 & 0 & xu & yu & zu
0 & 1 & wu & su & tu
\end{bmatrix}
\]  with \( (x, y, z, w, s, t) \in \{(0, 1, 0, 0, 1, 1), (1, 0, 0, 0, 1, 1), (0, 0, 0, 0, 1, 0), (1, 0, 1, 0, 1, 0), (0, 0, 0, 0, 0, 0), (1, 1, 1, 0, 0, 0), (1, 1, 0, 1, 1, 1), (1, 1, 1, 0, 1, 0), (1, 1, 0, 1, 1, 0), (0, 0, 0, 0, 1, 0), (1, 1, 1, 1, 1, 1), (0, 0, 1, 0, 1, 0)\};

- \[
\begin{bmatrix}
1 & 0 & xu & yu & zu
0 & 1 & 1 + wu & 1 + su & tu
\end{bmatrix}
\]  with \( (x, y, z, w, s, t) \in \{(0, 0, 1, 0, 0), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1, 1), (1, 0, 0, 1, 1, 0), (0, 0, 1, 1, 1, 0), (1, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0), (1, 1, 1, 0, 0, 0), (0, 1, 1, 0, 1, 0)\};

- \[
\begin{bmatrix}
1 & 0 & 1 + xu & yu & zu
0 & 1 & 1 + wu & 1 + su & 1 + tu
\end{bmatrix}
\]  with \( (x, y, z, w, s, t) \in \{(0, 1, 1, 0, 0, 0), (1, 1, 0, 1, 0), (0, 0, 1, 0, 1, 1), (0, 0, 1, 0, 1, 1)\};
\[
\begin{bmatrix}
1 & 0 & 1 + xu & 1 + yu & zu \\
0 & 1 & 1 + wu & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 0, 0, 1, 0), (1, 0, 1, 0, 1, 0), (0, 1, 1, 0, 0, 0), (1, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1, 0), (1, 0, 0, 1, 0, 1), (0, 1, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1), (1, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 1 + xu & 1 + yu & zu \\
0 & 1 & 1 + wu & 1 + su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 0, 1, 0, 1, 0), (0, 1, 1, 0, 1, 1), (0, 1, 1, 1, 0, 1), (1, 1, 0, 1, 1, 0)\};

\[
\begin{bmatrix}
1 & 0 & 1 + xu & yu & zu \\
0 & 1 & 1 + wu & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 0, 0, 0, 1), (0, 0, 1, 0, 1, 1), (0, 1, 1, 1, 0, 0), (0, 1, 1, 0, 1, 1), (1, 1, 0, 0, 1, 1), (1, 0, 1, 0, 0, 0), (1, 0, 1, 0, 1, 0), (0, 0, 0, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 1 + xu & 1 + yu & zu \\
0 & 1 & wu & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 1, 1, 0, 1), (1, 1, 0, 1, 1, 0), (0, 0, 1, 1, 1, 1), (0, 1, 0, 1, 0, 1), (0, 0, 1, 1, 0, 0), (1, 1, 1, 1, 1, 1), (0, 0, 1, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 1 + xu & yu & zu \\
0 & 1 & 1 + wu & 1 + su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(0, 1, 1, 1, 1, 0), (0, 0, 1, 0, 1, 1), (1, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 1), (0, 1, 1, 0, 1, 0)\};

\[
\begin{bmatrix}
1 & 0 & 1 + xu & 1 + yu & 1 + zu \\
0 & 1 & 1 + wu & 1 + su & 1 + tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 0, 1, 1, 0, 1), (1, 1, 1, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 0)\};
Some preliminaries
Mass formulae
Classification

Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
1 & 0 & 1 + xu & 1 + yu & 1 + zu \\
0 & 1 & 1 + wu & su & tu
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{(1, 1, 1, 1, 0, 1), (0, 0, 1, 0, 0), (0, 0, 1, 0, 0, 0), (1, 0, 1, 1, 0, 0), (1, 0, 0, 0, 0, 0)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & su & tu
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{(1, 1, 1, 1, 1), (1, 0, 1, 1, 0, 1), (0, 0, 0, 1, 1, 0), (0, 1, 1, 1, 0, 1), (1, 0, 1, 0, 1, 0), (1, 0, 1, 1, 0, 0), (1, 1, 1, 1, 1, 1), (1, 0, 1, 0, 0, 1)\};

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 1 + yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & su & 1 + tu
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{(0, 1, 1, 0, 0, 0), (0, 0, 0, 0, 1, 0), (1, 0, 1, 1, 1, 1)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & 1 + su & 1 + tu
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{(0, 0, 1, 0, 0, 1), (1, 1, 1, 1, 1, 1), (0, 1, 0, 1, 0, 1), (0, 0, 0, 0, 0, 0)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & 1 + su & 1 + tu
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{(0, 0, 0, 0, 0, 1), (0, 0, 1, 0, 1, 0), (1, 0, 1, 1, 0, 0), (0, 1, 1, 1, 1, 0), (1, 1, 1, 1, 0, 0)\}\);
Some preliminaries

Mass formulae

Classification

Self-orthogonal codes

Self-dual codes

LCD codes

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & su & 1 + tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (0, 1, 1, 0, 0), (1, 0, 0, 1, 1, 0), (1, 0, 0, 1, 0, 0), (0, 0, 1, 0, 0, 1) \}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & zu & 1 + wu \\
0 & 0 & 1 & su & tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (1, 0, 1, 1, 1, 0), (1, 1, 0, 1, 0, 0), (1, 0, 0, 0, 0, 0) \}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & 1 + su & 1 + tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (0, 1, 1, 0, 1), (1, 0, 1, 0, 1) \}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & 1 + su & 1 + tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (0, 1, 1, 1, 1, 0), (0, 0, 0, 1, 0, 1), (1, 1, 0, 0, 1, 1) \}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & 1 + zu & wu \\
0 & 0 & 1 & 1 + su & tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (1, 0, 0, 1, 0, 0), (0, 1, 1, 0, 1, 0), (1, 1, 0, 1, 0, 0), (1, 1, 0, 0, 0, 0) \}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 1 + yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & 1 + su & 1 + tu \\
\end{bmatrix}
\]

with \((x, y, z, w, s, t) \in \{ (0, 0, 0, 1, 1, 0), (0, 1, 0, 1, 0, 1), (1, 1, 1, 0, 1, 1) \}\);
Some preliminaries
Mass formulae
Classification

Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & yu \\
0 & 1 & 0 & 1 + zu & wu \\
0 & 0 & 1 & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 1), (0, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1, 1), (0, 1, 0, 0, 0, 1), (0, 0, 0, 0, 0, 1)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 1, 0, 1, 1)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & 1 + su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 1, 0, 1, 1), (1, 0, 1, 0, 0, 0), (1, 1, 0, 1, 1, 1), (0, 0, 1, 1, 1, 1), (1, 1, 1, 1, 1, 0), (0, 0, 1, 0, 0, 0)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 0, 1, 1, 1, 0), (0, 0, 0, 0, 1, 1), (0, 1, 1, 0, 0, 0)\}\);

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 1 + yu \\
0 & 1 & 0 & 1 + zu & 1 + wu \\
0 & 0 & 1 & su & tu
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 0, 0, 1, 0, 0)\}\);
Some preliminaries
Mass formulae
Classification
Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
1 & 0 & 0 & xu & 1 + yu \\
0 & 1 & 0 & zu & 1 + wu \\
0 & 0 & 1 & 1 + su & 1 + tu \\
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 0, 1, 0, 0, 1), (0, 0, 1, 1, 0, 1), (0, 0, 1, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + u & 1 \\
0 & 1 & 0 & u & 1 \\
0 & 0 & 1 & 0 & 1 \\
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & u \\
0 & 1 & 0 & 1 + u & 0 \\
0 & 0 & 1 & 1 & 1 \\
\end{bmatrix};
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & su & 1 + tu \\
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(0, 0, 0, 1, 0, 0), (1, 1, 0, 1, 1, 1), (0, 0, 1, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu & 1 + yu \\
0 & 1 & 0 & zu & wu \\
0 & 0 & 1 & su & tu \\
\end{bmatrix}
\]
with \((x, y, z, w, s, t) \in \{(1, 1, 0, 1, 1, 1), (0, 0, 0, 0, 0, 0)\};

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & xu \\
0 & 1 & 0 & 0 & yu \\
0 & 0 & 1 & 0 & zu \\
0 & 0 & 0 & 1 & wu \\
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(0, 1, 1, 1), (0, 0, 0, 0), (1, 1, 1, 1), (0, 1, 1, 0), (0, 0, 1, 0)\};

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 + xu \\
0 & 1 & 0 & 0 & 1 + yu \\
0 & 0 & 1 & 0 & 1 + zu \\
0 & 0 & 0 & 1 & 1 + wu \\
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(0, 1, 1, 0), (0, 0, 1, 0), (0, 0, 0, 0)\};
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & xu \\
0 & 1 & 0 & 0 & 1 + yu \\
0 & 0 & 1 & 0 & 1 + zu \\
0 & 0 & 0 & 1 & wu
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(0, 1, 1, 0), (0, 0, 0, 0)\}\

\[
I_5, \quad \begin{bmatrix}
1 & 0 & 0 & 0 & u \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 + u \\
0 & 0 & 0 & 1 & 1 + u
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & u \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 & u \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & u \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}.
Total number of LCD and inequivalent LCD codes of a given rank and length 4 over $\mathbb{F}_3[u]/<u^2>$

<table>
<thead>
<tr>
<th>Rank $k$</th>
<th>Total number of LCD codes of length 4 and rank $k$ over $\mathbb{F}_3[u]/&lt;u^2&gt;$</th>
<th>Number of inequivalent LCD codes of length 4 and rank $k$ over $\mathbb{F}_3[u]/&lt;u^2&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>648</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>7290</td>
<td>63</td>
</tr>
<tr>
<td>3</td>
<td>648</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
There are precisely 94 inequivalent non-zero LCD codes of length 4 over $\mathbb{F}_3[u]/<u^2>$, whose generator matrices are as listed below:

- $\begin{bmatrix} 1 & xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(2, 0, 0), (0, 1, 1), (2, 1, 1), (0, 0, 0)\}$;
- $\begin{bmatrix} 1 & 1 + xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0, 0, 2), (2, 1, 1), (0, 1, 1)\}$;
- $\begin{bmatrix} 1 & 2 + xu & yu & zu \end{bmatrix}$ with $(x, y, z) \in \{(0, 0, 0), (2, 2, 0), (1, 0, 0)\}$;
- $\begin{bmatrix} 1 & 1 + xu & 1 + yu & 2 + zu \end{bmatrix}$ with $(x, y, z) \in \{(1, 1, 1), (2, 2, 1), (0, 0, 0), (2, 0, 1)\}$;
- $\begin{bmatrix} 1 & 1 + u & 2 + 2u & 2 + 2u \end{bmatrix}$;
- $\begin{bmatrix} 1 & 0 & 2 + xu & 1 + yu \\ 0 & 1 & 1 + zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 2), (0, 2, 0, 1), (2, 1, 2, 0)\}$;
- $\begin{bmatrix} 1 & 0 & 1 + xu & 2 + yu \\ 0 & 1 & 1 + zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 2), (2, 1, 2, 0), (2, 1, 1, 0), (2, 0, 1, 0), (1, 0, 2, 1), (0, 0, 0, 0), (1, 1, 0, 0)\}$;
- $\begin{bmatrix} 1 & 0 & 1 + xu & 2 + yu \\ 0 & 1 & 1 + zu & 2 + wu \end{bmatrix}$ with $(x, y, z, w) \in \{(0, 1, 0, 1), (1, 0, 1, 0), (1, 2, 0, 0), (1, 1, 0, 0), (1, 0, 0, 2), (0, 2, 2, 2)\}$;
- $\begin{bmatrix} 1 & 0 & 1 + xu & yu \\ 0 & 1 & zu & wu \end{bmatrix}$ with $(x, y, z, w) \in \{(2, 1, 0, 0), (1, 0, 0, 0), (0, 1, 2, 0), (1, 1, 1, 0), (0, 1, 0, 1), (1, 2, 0, 1), (1, 2, 1, 1), (2, 1, 2, 1), (2, 0, 0, 1), (2, 0, 1, 1), (0, 0, 0, 2), (0, 0, 2, 0)\}$;
Some preliminaries
Mass formulae
Classification
Self-orthogonal codes
Self-dual codes
LCD codes

\[
\begin{bmatrix}
1 & 0 & 1 + xu & yu \\
0 & 1 & zu & 1 + wu
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(1, 1, 1, 0), (1, 1, 0, 0), (2, 0, 2, 0), (1, 0, 0, 0), (2, 2, 2, 2), (1, 1, 1, 2), (1, 1, 0, 2), (1, 0, 2, 0), (1, 2, 1, 1), (0, 2, 0, 0), (2, 0, 0, 2), (0, 2, 1, 0), (0, 0, 0, 0)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & 2 + xu & yu \\
0 & 1 & 1 + zu & 1 + wu
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(2, 1, 2, 0), (2, 2, 2, 0), (2, 1, 1, 0), (2, 2, 1, 0), (2, 2, 0, 2), (1, 2, 1, 2), (2, 1, 1, 2), (0, 0, 1, 0), (0, 0, 2, 0)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & xu & yu \\
0 & 1 & 1 + zu & wu
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(0, 0, 0, 0), (2, 2, 0, 0), (2, 2, 0, 1), (0, 0, 0, 1)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & xu & yu \\
0 & 1 & zu & wu
\end{bmatrix}
\]
with \((x, y, z, w) \in \{(2, 2, 0, 0), (0, 0, 0, 0), (1, 2, 1, 1), (1, 2, 1, 2), (2, 2, 2, 0), (1, 0, 2, 0), (0, 2, 0, 0), (0, 2, 2, 0)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 2 + u \\
0 & 1 & 2 + u & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 1 + u & u \\
0 & 1 & u & 1 + u
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & xu \\
0 & 1 & 0 & yu \\
0 & 0 & 1 & zu
\end{bmatrix}
\]
with \((x, y, z) \in \{(1, 1, 0), (2, 0, 0), (1, 1, 1), (0, 0, 0)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & 0 & xu \\
0 & 1 & 0 & 2 + yu \\
0 & 0 & 1 & zu
\end{bmatrix}
\]
with \((x, y, z) \in \{(1, 2, 1), (0, 2, 0), (2, 0, 1), (0, 0, 2)\} ;
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 + xu \\
0 & 1 & 0 & 2 + yu \\
0 & 0 & 1 & 1 + zu
\end{bmatrix}
\]
with \((x, y, z) \in \{(2, 1, 0), (0, 0, 1), (0, 0, 2)\} ;
\]
\[ I_4, \begin{bmatrix} 1 & 0 & 0 & 2 + u \\ 0 & 1 & 0 & 2 + 2u \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 + 2u \\ 0 & 1 & 0 & 2u \\ 0 & 0 & 1 & 0 \end{bmatrix}, \]

\[ \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \]
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Thank you...