A termording free variation of Möller algorithm

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Abstract

This paper is part of a series of articles in the context of Degröbnerization [5, 6, 2, 7, 8, 9] and is devoted to give the best version of the original *Möller Algorithm* [13] proceeding by induction on the points[1] of which we presented in ACA2023 [2] and ISSAC'22 [5] a version available for each ideal defined by (not necessarily commutative) functionals over any *effective ring*.

Gröbner bases's theory plays an important role in Computer Algebra and many applications have been solved by considering them as a preprocessing, and saying "if we have the Gröbner basis, then the problem is easily solved". This is undoubtedly true, but it does not take into account that finding a Gröbner basis is not always an easy task.

Luckily, there are practical problems for which Gröbnerian technology is not the only way to get a solution, and this allows us to switch to a new paradigm: *Degröbnerization* [2].

Such paradigm consists in

- using linear algebra and combinatorial methods instead of Gröbner basis computation and Buchberger's reduction and
- completely change perspective in the algebraic representation of our problems, substituting the Gröbnerian technology representation, based on *polynomial ideals*, to a representation given by *quotient algebras* expressed via a vector-space basis and multiplication (Auzinger-Stetter) matrices [1].

We recall that classical Möller Algorithm

- ◇ takes as input a set of functionals ordered in such a way that each initial segment defines a zero-dimensional ideal thus producing a Macaulay chain [17] of such ideal,
- ◇ which is easily produced for a 0-dimensional ideal of polynomials;

[1] in opposite to *Buchberger-Möller Algorithm* [15-17] which proceeds by induction on terms.

◇ produces for each ideal in the Macaulay chain not only its representation as a *quotient algebra* expressed via a vector-space and Auzinger–Stetter multiplication matrices [1],

◇ but also triangular and separator polynomials can be derived as well as the transformation matrix linking them

◇ and requires at most the evalauation of each such functional to each term needed to express the wanted vector-space basis.

The new aspect of this reformulation is that the present version is completely free with respect to term-orderings and can be applied using *any* total ordering (not necessarily a semigroup one) for ordering the terms needed to express the wanted results; actually we can apply it to *any* finite set $\mathbf{T}$ of terms with the only requirement that 1 be connected to $\mathbf{T}$ [18, 19, 20].

What oriented our investigation towards a version of the algorithm which at the same time does not require a semigroup ordering and that covers a wide class of algebras was our intention to apply Degröbnerization techniques in the context of Algebraic Statistics. This required us a careful reading of [3], which is the strongest supporter of the application of both Gröbnerian technology and Buchberger-Möller Algorithm toward Algebraic Statistics and where we read

- *Another class of statistical models we shall consider are linear models whose vector space basis is formed by polynomials which are not monomials*;
- *Example 7 is not a corner cut*[2] *model. However, it is the most symmetric of the models in the statistical fan. In fact, to destroy symmetry is a feature of Gröbner basis computation, as term orderings intrinsically do not preserve symmetries, which are often preferred in statistical models* .

Example 7 of [3] consists in considering as functionals the evaluation of the 5 ordered points

$$\mathcal{F} = \{(0,0),(1,-1),(-1,1),(0,1),(1,0)\} \subset \mathbb{Q}^2$$

and produce the desired data w.r.t. the algebra

$$\mathrm{Span}_{\mathbb{Q}}\, \mathbf{T} \equiv \mathcal{P}/\mathbb{I}(\mathcal{F}).$$

where we are denoting $\mathcal{P} := \mathbb{Q}[x_1, x_2]$,

$$\mathbb{I}(\mathcal{F}) := \{f(x_1,x_2) \in \mathcal{P} : f(a,b) = 0, (a,b) \in \mathcal{F}\},$$

$\mathbf{T}$ the ordered set of terms

$$\mathbf{T} := \{1, x_1, x_2, x_1^2, x_2^2\}.$$

Our algorithm produces the desired symmetric basis $\mathcal{B} = \{x^3 - x, x^2y - 1/2x^2 + 1/2x + 1/2y^2 - 1/2y, y^3 - y, xy - 1/2x + 1/2y^2 - 1/2y + 1/2x^2, xy^2 - 1/2x - 1/2y^2 + 1/2y + 1/2x^2\}$.

[2] Recall that a Monomial Basis of a 0-dimensional ideal of polynomials which is *Hierarchical, i.e.* an order ideal, is called a *Corner Cut* when it is the Gröbner escalier/normal set modulo the Gröbner basis of such ideal w.r.t. a term ordering.

Keywords
Möller algorithm, Algebraic Statistics Degröbnerization

References

[1] Auzinger W., Stetter H.J., *An Elimination Algorithm for the Computation of all Zeros of a System of Multivariate Polynomial Equations*, I.S.N.M.86 (1988), 11–30, Birkhäuser

[2] Ceria M., Lundqvist S., Mora T., *Degrb̈nerization: a political manifesto*, Applicable Algebra in Engineering, Communication and Computing, **33** (6), 675 – 723, (2022).

[3] Pistone G., Riccomagno E., Rogantin M.P., *Methods in Algebraic Statistics for the Design of Experiments*. In: Pronzato, L., Zhigljavsky, A. (eds) Optimal Design and Related Areas in Optimization and Statistics. Springer Optimization and Its Applications, **28**, Springer, New York, NY, (2009).

[4] Ceria M., Mora T., Visconti A., *Degroebnerization for data modelling problems*, abstract for ACA23.

[5] Ceria M., Mora T., *A Degroebnerization Approach to Algebraic Statistics*, ISSAC '22: Proceedings of the 2022 International Symposium on Symbolic and Algebraic ComputationJuly 2022Pages 419–428

[6] Ceria, M., Lundqvist, S., Mora, T., *Degröbnerization and Its Applications: Reverse Engineering of Gene Regulatory Networks* CEUR Workshop Proceedings of the 6th International Workshop on Satisfiability Checking and Symbolic Computation, Vol.3273, p. 27-30, 2022.

[7] Mora T. slides available at
https://drive.google.com/file/d/1NlbiEehGGWIbWcbsypYNFY0oknpexpbL/view?usp=sharing

[8] Mora T. slides available at
https://drive.google.com/file/d/1ye4P7WrBphbRk1S1ncbdBxgxxPFb1IWw/view?usp=sharing

[9] slides available at
https://drive.google.com/file/d/1QKobQNLFlvmMtX6n-9ZPyjG382dVKdJ0/view?usp=sharing

[10] Cerlienco L., Mureddu M., *Algoritmi combinatori per l'interpolazione polinomiale in dimensione* ≥ 2, Publ. I.R.M.A. Strasbourg, 1993, 461/S-24 Actes 24^e Séminaire Lotharingien, p.39-76.

[11] Cerlienco L., Mureddu M., *From algebraic sets to monomial linear bases by means of combinatorial algorithms*, Discrete Math. 139, 73 – 87.

[12] Cerlienco L., Mureddu M., *Multivariate Interpolation and Standard Bases for Macaulay Modules*, J. Algebra 251 (2002), 686 − 726.

[13] Marinari, M.G., Möller, H.M., Mora, T., *Gröbner bases of ideals defined by functionals with an application to ideals of projective points*, Applicable Algebra in Engineering, Communication and Computing 4(2), 103–145 (1993)

[14] Buchberger B., Möller H.M., *The construction of multivariate polynomials with preassigned zeros* , European Computer Algebra Conference, 24–3, Springer, 1982.

[15] Mora T., Robbiano L., *Computational Algebraic Geometry and Commutative Algebra, Cortona-91*, 34, 106–150, 1993, Cambridge University Press.

[16] Abbott J., Bigatti A, Kreuzer M., Robbiano L., *Computing ideals of points*, Journal of Symbolic Computation, 30(4), 341–356, Elsevier, 2000

[17] M.E. Alonso, M.G. Marinari, M.T. Mora, *The Big Mother of All the Dualities, II: Macaulay Bases.* J. AAECC **17**, 409–451 (2006).

[18] Mourrain B. *A New Criterion for Normal Form Algorithms.* In: Fossorier M., Imai H., Lin S., Poli A. (eds) Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. AAECC 1999. Lecture Notes in Computer Science, vol 1719. Springer, Berlin, Heidelberg (1999)

[19] Mourrain B., *Bezoutian and quotient ring structure* J. Symb. Comp. **39** (2005), 397-415

[20] Mourrain B., Trebuchet P., *Solving projective complete intersection faster* , Proc. ISSAC'00 (2000), 234–241, ACM