Semigroup collaborations between elementary operations

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a joint work with

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Abstract

Given two operations $*$ and $\circ$ on a set S, an operation $\star$ on S is said to be a *collaboration* between $*$ and $\circ$ if for all $a, b \in S$, $a \star b \in \{a * b, a \circ b\}$. Another term for collaborations is two-option operations.

We are interested in learning what associative collaborations of two given operations $*$ and $\circ$ there may be. We do not require that $*$ and $\circ$ themselves be associative. As an initial experiment, we consider *Plus-Minus* operations (i.e. collaborations between addition and subtraction on an abelian group) and *Plus-Times* operations (i.e. collaborations between the addition and multiplication operations on a semiring.) Our study of Plus-Minus operations focuses on the additive integers but extends to ordered groups. For Plus Times operations, we make some headway in the case of the semiring of natural numbers. We produce an exhaustive list of associative collaborations between the usual addition and multiplication on the natural numbers.

The Plus-Times operations we found are all examples of a type of construction which we define here and that we call *augmentations by multidentities*. An augmentation by multidenties combines two separate magmas A and B to create another, $A(B)$, having $A \sqcup B$ as underlying set, and in such a way that the elements of B act as identities over those of A. Hence, B consists of a sort of multiple identities (explaining the moniker multidentities.) When A and B are both semigroups then so is $A(B)$. Understanding the connection between certain collaborations and augmentation by multidenties removes, in several cases, the need for cumbersome computations to verify asssociativity.

In closing, we discuss connections between group collaborations and skew braces.

Keywords Semigroups, semirings, braces, two-option operation, group collaboration, augmentation by multidenties.

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