A termording free variation of Möller algorithm

NCRA VIII
Lens

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This conference is part of a series of articles and conferences in the context of Degröbnerization prepared with Michela Ceria, Samuel Lundqvist and Andrea Visconti.

Degröbnerization was introduced for the first time in 2010 into a course at Trento's Cryptolab, implicitly in a commutative setting, but later explicitly in a non-commutative settings at ACA2018 and UMI2019 and was definitely formalized in a conference at ACA2021.

Degröbnerization

Gröbner bases's theory plays an important role in Computer Algebra and many applications have been solved by considering them as a preprocessing, and saying ”if we have the Gröbner basis, then the problem is easily solved”. This is undoubtedly true, but it does not take into account that *finding a Gröbner basis is not always an easy task*. The computation can become computationally hard and there are cases in which it is even infeasible.

Degröbnerization

In short the approach consists in finding new ways to solve practical problems that have been originally solved using Gröbner basis computation and Buchberger's reduction, leaving the use of the latter only to the cases where it is really necessary.
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In short the approach consists in finding new ways to solve practical problems that have been originally solved using Gröbner basis computation and Buchberger's reduction, leaving the use of the latter only to the cases where it is really necessary. Degröbnerization consists in using linear algebra and combinatorial methods and avoiding the concept of Gröbner bases, in favourof methods relying on linear algebra and combinatorics.

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In short the approach consists in finding new ways to solve practical problems that have been originally solved using Gröbner basis computation and Buchberger's reduction, leaving the use of the latter only to the cases where it is really necessary. Degröbnerization consists change perspective in the algebraic representation of our problems, substituting the prior representation, based on *polynomial ideals* to a representation given by *quotient algebras*, expressed via a vector-space basis and multiplication (Auzinger-Stetter) matrices

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Political Issue

Our variation

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- but also triangular and separator polynomials can be derived, as in the original version, as well as the transformation matrix linking them and
- requires at most the evaluation of each such functional to each term needed to express the wanted vector-space basis .

Setting

$$\mathcal{T} := \{x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma_1, ..., \gamma_n \in \mathbb{N}\} \subset \mathcal{S} = \langle x_1, ..., x_n \rangle$$

ordered by a total ordering (not necessarily a semigroup one)
$\mathcal{P} := \mathbf{k}[x_1, ..., x_n] = \mathrm{Span}_{\mathbf{k}} \mathcal{T}$ and $\mathcal{Q} := \mathbf{k}\langle x_1, ..., x_n \rangle = \mathrm{Span}_{\mathbf{k}} \mathcal{S} \supseteq \mathcal{P}$.
An effective ring $\mathfrak{A}$ given as a $\mathbf{k}$-submodule of either $\mathcal{P}$ or $\mathcal{Q}$ via an ordered subset $\mathcal{U}$ of terms of either $\mathcal{T}$ or $\mathcal{S}$

A finite (not necessarily linearly indipendet) set
$\mathbb{L} = \{L_i, 1 \leq i \leq s\} \subset \mathrm{Hom}_{\mathbf{k}}(\mathcal{U}, \mathbf{k})$ of $\mathbf{k}$-linear fuctionals $L_i : \mathcal{U} \to \mathbf{k}$

Aim

For $\mathsf{I} := \{f \in \mathcal{U} : L_i(f) = 0\}$ our combinatorial tools return an order ideal $\mathbf{N}$ such that $\mathcal{U}/\mathsf{I} \cong \mathrm{Span}_{\mathbf{k}}\, \mathbf{N}$ where

$$\#(\mathbf{N}(\mathsf{I})) = \deg(\mathsf{I}) = \dim_k(\mathbb{L}) =: r \leq s = \#\mathbb{L}.$$

Aim

For $\mathsf{I} := \{f \in \mathcal{U} : L_i(f) = 0\}$ produce

an integer $r \in \mathbb{N}$,

an order ideal $\mathbf{N} := \{t_1, \ldots, t_r\} \subset \mathcal{T}$,

an ordered subset $\Lambda := \{\lambda_1, \ldots, \lambda_r\} \subset \mathbb{L}$

an ordered set $\mathbf{q} := \{q_1, \ldots, q_r\} \subset \mathcal{P}$

such that it holds:

$r = \deg(\mathsf{I}) = \dim_k(\mathbb{L})$,

$\mathbf{N}(\mathsf{I}) = \mathbf{N}$,

$\mathrm{Span}_k(\Lambda) = \mathrm{Span}_k(\mathbb{L})$,

$\mathrm{Span}_k\{t_1, \ldots, t_\sigma\} = \mathrm{Span}_k\{q_1, \ldots, q_\sigma\}, \forall \sigma \leq r$,

$\{q_1, \ldots, q_\sigma\}$, $\{\lambda_1, \ldots, \lambda_\sigma\}$ are triangular, $\forall \sigma \leq r$.

Tool

$\mathbb{L} = \{L_i, 1 \le i \le s\} \subset \mathrm{Hom}_{\mathbf{k}}(\mathcal{U}, \mathbf{k})$, $\mathsf{I} := \{f \in \mathcal{U} : L_i(f) = 0\}$
$\mathbf{N}(\mathsf{I}) = \{t_1, \dots, t_r\}$
Consider the $s \times r$ matrix $\ell_i(t_j)$ *whose columns* are the vectors $v(t_j, \mathbb{L})$ and *are linearly independent*, since any relation $\sum_j c_j v(t_j, \mathbb{L}) = 0$ would imply

$$\ell_i(\sum_j c_j t_j) = \sum_j c_j \ell_i(t_j) = 0 \text{ and } \sum_j c_j t_j \in \{f \in \mathcal{U} : L_i(f) = 0\} = \mathsf{I}$$

contradicting the definition of $\mathbf{N}(\mathsf{I})$.
The matrix $\ell_i(t_j)$ has rank $r \le s$ and it is possible to extract an ordered subset

$$\Lambda := \{\lambda_1, \dots, \lambda_r\} \subset \mathbb{L}, \quad \mathrm{Span}_{\mathbf{k}}\{\Lambda\} = \mathrm{Span}_{\mathbf{k}}\{\mathbb{L}\}$$

and to re-enumerate the terms in $\mathbf{N}(\mathsf{I})$ in such a way that *each principal minor* $\lambda_i(t_j), 1 \le i, j \le \sigma \le r$ *is invertible*.

TOOL:BORDER

The *border* of $\mathbf{N}$ is the set of terms $\mathbf{B} := \{x_i t : t \in \mathbf{N}\} \setminus \mathbf{N}$. With this notation, the related border bases are the sets $\mathcal{B}\{t - \mathrm{Nf}(t) : t \in \mathbf{B}\}$ and $\mathcal{A} := \{t - \mathrm{Nf}(t) : t \in \mathbf{B} \cup \mathbf{N}\}$ where Nf(t) is the normal form of t, the only polynomail $t - \sum_{\tau \in \mathbf{N}(\mathsf{I})} c_\tau \tau \in \mathsf{I}$.

Pistone G., Riccomagno E., Rogantin M.P., **Methods in algebraic statistics for the design of experiments** in **Optimal Design and Related Areas in Optimization and Statistics**, 97–132, 2009, Springer

[1] Corner cut: a Hierarchical monomial basis which is the normal set-escalier related to a Gröbner basis

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Another class of statistical models we shall consider are linear models whose vector space basis is formed by polynomials v_j which are not monomials

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Another class of statistical models we shall consider are linear models whose vector space basis is formed by polynomials v_j which are not monomials In Example 7 we show that the model $1, x_1, x_1^2, x_2, x_2^2$ is not a corner cut [1] model.

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Pistone G., Riccomagno E., Rogantin M.P., **Methods in algebraic statistics for the design of experiments** in **Optimal Design and Related Areas in Optimization and Statistics**, 97–132, 2009, Springer

> **Another class of statistical models we shall consider are linear models whose vector space basis is formed by polynomials v_j which are not monomials** In Example 7 we show that the model $1, x_1, x_1^2, x_2, x_2^2$ is not a corner cut [1] model. owever, it is the most symmetric of the models in the statistical fan. In fact, to distroy symmetry is a feature of Gröbner basis computation, as term orderings intrinsically do not preserve symmetries, which are often preferred in statistical models

That's why you should never think of using Gröbner bases in Algebraic Statistics

[1] Corner cut: a Hierarchical monomial basis which is the normal set-escalier related to a Gröbner basis

Example

$\mathcal{P} := \mathbb{Q}[x, y]$.
The design ideal $\mathbb{I}(\mathcal{F})$ with

$$\mathcal{F} = \{(0, 0), (1, -1), (-1, 1), (0, 1), (1, 0)\}$$

and the Hierarchical monomial basis $\{1, x_1, x_1^2, x_2, x_2^2\}$.

EXAMPLE

$\mathcal{P} := \mathbb{Q}[x, y]$.
Let us consider the five points

$$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1), P_5 = (1,0)$$

and the related set of functionals $\mathbb{L} = \{L_1, ..., L_5\}$ such that L_i is the evaluation at P_i, for $i = 1, ..., 5$.
$\mathbb{L} = \{L_i, 1 \leq i \leq s\} \subset \mathrm{Hom}_{\mathbb{Q}}(\mathcal{P}, \mathbb{Q})$, $\mathsf{I} := \{f \in \mathcal{P} : L_i(f) = 0\}$
$\mathbf{N}(\mathsf{I}) = \{t_1, \ldots, t_r\}$.

Algorithm:Output

The algorithm is iterative on each point/functional. At the step i it considers the data for the ideal $\mathrm{I}_i := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j \leq i\}$, take the point P_{i+1} and return the same data for $\mathrm{I}_{i+1} := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j < i\}$

the associated escalier $\mathbf{N}(\mathbb{L}_5) = \{t_1, t_2, t_3, t_4, t_5\}$;

$\mathbf{N}(\mathbb{L}_5) = \{t_1 = 1, t_2 = xt_1 = x, t_3 = xt_2 = x^2, t_4 = yt_1 = y, t_5 = yt_4 = y^2\}$

$P_1 = (0, 0), P_2 = (1, -1), P_3 = (-1, 1), P_4 = (0, 1)$

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The algorithm is iterative on each point/functional. At the step i it considers the data for the ideal $\mathrm{I}_i := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j \leq i\}$, take the point P_{i+1} and return the same data for $\mathrm{I}_{i+1} := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j < i\}$

$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1)$

the current border bases $\mathcal{A}$

$\mathcal{A} = \{v_1 = 1, v_2 = x, v_3 = x^2 - x = xv_2 - v_2, v_4 = y + x = yv_1 + v_2, v_5 = y^2 - y - x - x^2, xv_4 - v_4 - v_5/2 = 1/2x^2 + xy - 1/2x + 1/2y^2 - 1/2y = yv_2 + v_2 + v_3 + v_5/2, xv_3 + v_3 = x^3 - x, xv_5 - v_5 = -x^3 + xy^2 - xy + x - y^2 + y, yv_3 - v_3 = x^2y - xy - x^2 + x, yv_4 - v_4 = y^2 + xy - x - y, yv_5 = y^3 - x^2y - xy - y^2\}$

Algorithm:Output

The algorithm is iterative on each point/functional. At the step i it considers the data for the ideal $\mathrm{I}_i := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j \leq i\}$, take the point P_{i+1} and return the same data for
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the current border bases $\mathcal{A}$ and $\mathcal{B}$

$\mathcal{A} = \{v_1 = 1, v_2 = x, v_3 = x^2 - x = xv_2 - v_2, v_4 = y + x = yv_1 + v_2, v_5 = y^2 - y - x - x^2, xv_4 - v_4 - v_5/2 = 1/2x^2 + xy - 1/2x + 1/2y^2 - 1/2y = yv_2 + v_2 + v_3 + v_5/2, xv_3 + v_3 = x^3 - x, xv_5 - v_5 = -x^3 + xy^2 - xy + x - y^2 + y, yv_3 - v_3 = x^2y - xy - x^2 + x, yv_4 - v_4 = y^2 + xy - x - y, yv_5 = y^3 - x^2y - xy - y^2\}$

,

$\mathcal{B} = \{x^3 - x, x^2y - 1/2x^2 + 1/2x + 1/2y^2 - 1/2y, y^3 - y, xy - 1/2x + 1/2y^2 - 1/2y + 1/2x^2, xy^2 - 1/2x - 1/2y^2 + 1/2y + 1/2x^2\}.$

ALGORITHM:OUTPUT

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$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1)$

a basis for the quotient algebra modulo I_5, i.e. $\mathrm{V}(\mathbb{L}_5) := \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$;

$v_1 = 1, v_2 = x, v_3 = x^2 - x = xv_2 - v_2, v_4 = y + x = yv_1 + v_2, v_5 = y^2 - y - x - x^2$

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$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1)$

the Auzinger-Stetter matrices;

$A_x =$

	v_1	v_2	v_3	v_4	v_5
$v_1 = 1$	0	1	0	0	0
$v_2 = x$	0	1	1	0	0
$v_3 = x^2 - x$	0	0	-1	0	0
$v_4 = y + x$	0	0	0	0	$-1/2$
$v_5 = y^2 - y - x - x^2$	0	0	0	0	1

,

$A_y =$

	v_1	v_2	v_3	v_4	v_5
$v_1 = 1$	0	-1	0	1	0
$v_2 = x$	0	-1	-1	0	$-1/2$
$v_3 = x^2 - x$	0	0	1	0	0
$v_4 = y + x$	0	0	0	1	0
$v_5 = y^2 - y - x - x^2$	0	0	0	0	0

Algorithm:Output

The algorithm is iterative on each point/functional. At the step i it considers the data for the ideal $\mathrm{I}_i := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j \leq i\}$, take the point P_{i+1} and return the same data for
$\mathrm{I}_{i+1} := \{f \in \mathcal{P} : f(P_j) = 0, 1 \leq j < i\}$
$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1)$

a triangular set $\mathfrak{T}$ of polynomials;

$\mathfrak{T} = \{w_1 = v_1 = 1, w_2 = v_2 = x, w_3 = v_3/2 = (x^2 - x)/2, w_4 = v_4 = y + x, w_5 = -v_5/2 = -\frac{y^2-y-x-x^2}{2}\}$

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$P_1 = (0,0), P_2 = (1,-1), P_3 = (-1,1), P_4 = (0,1)$

a separator family $\mathfrak{S} = \{s_1 = v_1 - v_2 - v_3 - v_4 - v_5/2 = -1/2 * x^2 - (1/2)x - (1/2)y^2 - (1/2)y + 1, s_2 = v_2 + v_3/2 + v_5/2 = (1/2)y^2 - (1/2)y, s_3 = v_3/2 = \frac{x^2-x}{2}, s_4 = v_4 + v_5/2 = -(1/2)x^2 + (1/2)x + (1/2)y^2 + (1/2)y, s_5 = -v_5/2 = -\frac{y^2-y-x-x^2}{2}\}$

$$\Lambda := \{\lambda_1, \ldots, \lambda_r\} \subset \mathbb{L} = \{L_i, 1 \le i \le s\}$$

Finally column reduction allows to linearly express in terms of Λ the elemens $L_i \in \mathbb{L} \setminus \Lambda$

	L_1 (0,0)	L_2 (1,0)	L_3 (−1,0)	L_4 (0,1)	L_5 (1,1)	L_6 (−1,1)	L_7 (0,−1)	L_8 (1,−1)	L_9 (−1,−1)
v_1	**1**	1	1	1	1	1	1	1	1
v_2	0	**1**	−1	0	1	−1	0	1	−1
v_3	0	0	**1**	0	0	1	0	0	1
v_4	0	0	0	**1**	1	1	−1	−1	−1
v_5	0	0	0	0	**1**	−1	0	−1	1
v_6	0	0	0	0	0	**1**	0	0	−1

$$0 = L_7 + L_4 - 2L_1 = L_8 + L_5 - 2L_2 = L_9 + L_6 - 2L_3$$

Warning: Connected-to-1

Let us consider $\mathbf{k}[x, y]$ and denote $t = xy$. All terms of this ring will be uniquely reprensented in the form $x^i t^j y^k : ik = 0$.
Thus $x^i t^j y^k \leq x^a t^b y^c$ if

$i + k < a + c$ or

$i + k = a + c$ and $j < b$ or

$i + k = a + c$ and $j = b$ and $i > c$.

So that we have $1 < xy < x^2y^2 < \ldots < t^j < t^{j+1} < \ldots x < x^2y < \ldots < x^{j+1}y^j < \ldots y < xy^2 < \ldots < x^j y^{j+1} < \ldots$
Let us consider the following point set
$\mathbf{X} = \{(1, 0), (0, 1), (1, 1), (0, 0)\}$.

Warning: Connected-to-1

$P_1 = (1,0), P_2 = (0,1), P_3 = (1,1), P_4 = (0,0)$

	L_1 $(1,0)$	L_2 $(0,1)$	L_3 $(1,1)$	L_4 $(0,0)$
$v_1 = x$	**1**			
$x^2 - x$	0	0	0	0
xy	0	0		
$v_2 = y$	0	**1**		
$y^2 - y$	0	0	0	0
$v_3 = xy$	0		**1**	
$x^2y - xy$	0	0	0	0
$xy^2 - xy$	0	0	0	0

Connected to E

Let $\mathcal{V} \subset \mathcal{P}$ and denote

$\mathcal{V}^+ := \{\bar{v}_0 + \sum_{i=1}^n x_i \bar{v}_i, \bar{v}_i \in \mathcal{V}, 0 \leq i \leq n\}$,

for each $d \in \mathbb{N} \setminus \{0\}$ set $\mathcal{V}^{[d]} = \left(\mathcal{V}^{[d-1]}\right)^+$ starting from

$\mathcal{V}^{[0]} = \mathrm{Span}_{\mathbf{k}}(\mathcal{V})$,

$\mathcal{V}^{[*]} := \bigcup_{d \geq 0} \mathcal{V}^{[d]}$

$\mathcal{V}^{[*]}$ coincides with the ideal generated by $\mathcal{V}$.

Definition (Mourrain)

A vector space $\mathcal{V} \subset \mathcal{P}$ is said to be *connected* to $\bar{e} \in \mathcal{V}$ if, denoting $\mathcal{E} := \mathrm{Span}_{\mathbf{k}}\{\bar{e}\}$, for each $\bar{v} \in \mathcal{V} \setminus \mathcal{E}$, there exists $l > 0$ such that $\bar{v} \in \mathcal{E}^{[l]}$ and $\bar{v} = \bar{v}_0 + \sum_{i=1}^n x_i \bar{v}_i$, with $\bar{v}_i \in \mathcal{E}^{[l-1]} \cap \mathcal{V}, 0 \leq i \leq n$. □

Connected to E

Let $\mathcal{V} \subset \mathcal{P}$ and denote

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Theorem

If $\mathcal{V}$ is connected to $\bar{e}$, each element of $\mathcal{V}$ satisfies any property if

it is satisfied by $\bar{e}$ and

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If $\mathcal{V}$ is connected to $\bar{e}$, each element of $\mathcal{V}$ satisfies any property if

- *it is satisfied by $\bar{e}$ and*
- *it is satisfied by each linear combination of elements on $\mathcal{V}$ which satisfies it.*

Assume that the property is satisfied by each $\bar{v} \in \mathcal{E}^{[l-1]}$ and let $\bar{w} \in \mathcal{E}^{[l]}$. Since $\mathcal{V}$ is connected to $\bar{e}$, we have $\bar{w} = \bar{v}_0 + \sum_{i=1}^{n} x_i \bar{v}_i, \bar{v}_i \in \mathcal{E}^{[l-1]}, 0 \leq i \leq n$. and, by linearity, the property is satisfied also by $\bar{w} \in \mathcal{E}^{[l]}$. The claim then follows by induction.

Warning: Connected-to-1

$P_1 = (1,0), P_2 = (0,1), P_3 = (1,1), P_4 = (0,0)$ $E = \{x, y\}$ 1 is not connected to E whose elements vanish in P_4

A finite (not necessarily linearly indipendet) set $\mathbb{L} = \{L_i, 1 \leq i \leq s\} \subset \mathrm{Hom}_{\mathbf{k}}(\mathcal{P}, \mathbf{k})$ $\dim_k(\mathbb{L}) = r \leq s$

An ordered subset $\mathcal{U} = \{u_1, \ldots, u_r\}$ of terms of either $\mathcal{T}$ or $\mathcal{S}$, $\mathcal{U} = r$ such that 1 be connected to $\mathcal{U}$

$\mathbf{1} = \sum_{j=1} a_j u_j$

Warning: Connected-to-1

$P_1 = (1,0), P_2 = (0,1), P_3 = (1,1), P_4 = (0,0)$ $E = \{x, y\}$ 1 is not connected to E whose elements vanish in P_4

A finite (not necessarily linearly indipendet) set $\mathbb{L} = \{L_i, 1 \leq i \leq s\} \subset \mathrm{Hom}_{\mathbf{k}}(\mathcal{P}, \mathbf{k})$ $\dim_k(\mathbb{L}) = r \leq s$

An ordered subset $\mathcal{U} = \{u_1, \dots, u_r\}$ of terms of either $\mathcal{T}$ or $\mathcal{S}$, $\mathcal{U} = r$ such that 1 be connected to $\mathcal{U}$

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$$\bar{w} = \bar{v}_0 + \sum_{h=1}^{n} x_h \bar{v}_h, \bar{v}_h = \sum_{j=1} a_{hj} u_j$$

$$\implies \bar{w} = \sum_{j=1} \left(a_{0j} + \sum_{h=1}^{n} a_{hl} \sum_{j=1}^{i} \alpha_{hlj} \right) v_j$$

Thank you!