The structure of pairs

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Based on work in collaboration with Akian, Gaubert, Gatto, Jun, and Mincheva.

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Thanks to Professor Leroy for putting in the effort to bring about this event, and for inviting me to participate.

Overview

This is part of an ongoing project to develop the most general viable algebraic structure theory which includes classical algebra and idempotent (additive) semigroups. We broached the subject a few months ago in the 14th Ukraine Algebra Conference, which focused on linear algebra. Today we also want to discuss structure theory, for example the analog of the prime spectrum.

Assume that $\mathcal{T}$ is a given set.

1. A (left) **module** over $\mathcal{T}$, or $\mathcal{T}$-module, is an additive semigroup $(\mathcal{A}, +, 0)$ together with a (left) $\mathcal{T}$-action $\mathcal{T} \times \mathcal{A} \to \mathcal{A}$ (denoted as concatenation), for which
   1. 0 is absorbing, i.e. $a0 = 0$, for all $a \in \mathcal{T}$.
   2. The action is **distributive** over $\mathcal{T}$, in the sense that

$$a(b_1 + b_2) = ab_1 + ab_2, \quad \text{for all } a \in \mathcal{T},\ b_i \in \mathcal{A}.$$

We assume throughout that $\mathcal{T} \subseteq \mathcal{A}$, and $\mathcal{T}_0 := \mathcal{T} \cup \{0\}$ is a monoid containing a unit element 1 such that $1b = b$, and $(a_1 a_2)b = a_1(a_2 b)$ for all $a_i \in \mathcal{T}$ and $b \in \mathcal{A}$.

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A $\mathcal{T}$-module $\mathcal{A}$ is **admissible** if the semigroup $(\mathcal{A}, +, 0)$ is $\mathcal{T}$-spanned.

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3. A pair $(\mathcal{A}, \mathcal{A}_0)$ is **cancellative** if it satisfies the following two conditions for $a \in \mathcal{T}$, $b \in \mathcal{A}$:
   1. If $ab \in \mathcal{A}_0$, then $b \in \mathcal{A}_0$.
   2. If $ab_1 = ab_2$, then $b_1 = b_2$.

Motivation

$\mathcal{A}_0$ is a distinguished subset which takes the place of 0. Usually $\mathcal{A}_0$ is closed under addition. To have a robust theory we need one more property as a consolation for lack of negation.

Property N:

- There is an element $1^\dagger \in \mathcal{T}$, not necessarily unique, such that $1 + 1^\dagger = 1^\dagger + 1 \in \mathcal{A}_0$. In this case we define $e := 1 + 1^\dagger \in \mathcal{A}_0$.
- $ae = a + b$ for each $b \in \mathcal{T}$ such that $a + b \in \mathcal{A}_0$.

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We define the product $be = \sum a_i e$ for $b = \sum a_i$, $a_i \in \mathcal{T}$; surprisingly, this is well-defined, so $\mathcal{A}$ becomes a module over $\mathcal{T} \cup \mathcal{T} e$.
Lemma: $e^2 = e + e$.

Some kinds of pairs

1. A **semiring** $(\mathcal{A}, +, \cdot, 0, 1)$ satisfies all the properties of a ring (including associativity and distributivity of multiplication over addition), but without negation. We shall denote multiplication by concatenation, and assume that semirings have a 0 element that is additively neutral and also is multiplicatively absorbing, and have a unit element 1.
2. An **nd-semiring** satisfies all of the properties of a semiring except distributivity.
3. An **nd-semiring pair** is a pair $(\mathcal{A}, \mathcal{A}_0)$ for which $\mathcal{A}$ is an nd-semiring.

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   Warning: In this case e^2 under the nd-semiring multiplication need not match the previous definition.

One can formulate many classical concepts in these terms

1. In classical algebra, one could take $\mathcal{A}_0 = \mathcal{T} = \mathcal{A}$.
2. In any Abelian group $(\mathcal{A}, +)$ spanned by some distinguished set $\mathcal{T}$ containing 1, one can take $\mathcal{A}_0 = 0$.

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3. The **exterior pair**, where $\mathcal{A}$ is the tensor algebra, whose multiplication is written as $\wedge$ and $\mathcal{A} = \{v \wedge v, v \wedge w + w \wedge v : v, w \in \mathcal{A}\}$.

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4. $\mathcal{T}$ is an arbitrary cancellative monoid, $\mathcal{A} = \mathcal{T}_0 \cup \{\infty\}$, $\mathcal{A}_0 = \{0, \infty\}$, and $b_1 + b_2 = \infty$ for all $b_1 \neq b_2$ in $\mathcal{T} \cup \{\infty\}$. We call this the $\mathcal{A}_0$-**minimal pair**. There are two kinds:
    - First kind. Here $a + a = \infty$ for all $a \in \mathcal{T}$.
    - Second kind. Here $a + a = a$ for all $a \in \mathcal{T}$.

Polynomials over a semiring pair $(\mathcal{A}, \mathcal{A}_0)$ are a semiring pair $(\mathcal{A}[\Lambda], \mathcal{A}_0[\Lambda])$ (and we can take $\mathcal{T}_{\mathcal{A}[\Lambda]}$ to be the monomials with coefficients in $\mathcal{T}$).

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Although usually we take $(\mathcal{A}, \mathcal{A}_0)$ commutative, one also can take the matrix pair $(M_n(\mathcal{A}), M_n(\mathcal{A}_0))$, over $\cup_{i,j}\mathcal{T} e_{i,j} \cup \{0\}$.

Another example: Supertropical algebra

We start with a multiplicative monoid $\mathcal{T}_0$ with an absorbing element 0. The **standard supertropical semiring** is a quadruple $(\mathcal{A}, \mathcal{T}, \mathcal{G}_0)$ where $\mathcal{G}_0 \subset \mathcal{A}$ is an ordered submonoid with minimal element 0, $\mathcal{A} := \mathcal{T} \cup \mathcal{G}$, identifying the 0 of $\mathcal{T}_0$ and $\mathcal{G}_0$, with a projection $\nu : \mathcal{A} \to \mathcal{G}_0$, restricting to an isomorphism $\mathcal{T}_0 \to \mathcal{G}_0$. $\mathcal{A}$ is viewed as a semiring with the following operations, writing a° for $\nu(a)$:

$$a + b = \begin{cases} a & \text{whenever} \quad a^\circ > b^\circ, \\ b & \text{whenever} \quad a^\circ < b^\circ, \\ a^\circ & \text{whenever} \quad a^\circ = b^\circ. \end{cases}$$

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$\mathcal{A}_0 := \mathcal{G}$ is a semiring ideal of $\mathcal{A}$, and the pair $(\mathcal{A}, \mathcal{A}_0)$ is of the first kind.

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One reason that one may prefer the standard supertropical semiring in tropical algebra to the max-plus (or min-plus) algebra is that valuations satisfy $v(xy) = v(x) + v(y)$ whereas $v(x + y) = \{\min v(x), v(y)\}$ when $v(x) \neq v(y)$ but is ambiguous when $v(x) = v(y)$.

Another example: Pairs of hyper-semirings

The following notion dates back to F. Marty (1934) and M. Krasner (1935). $\mathcal{P}^*$ denotes the set of nonempty subsets.

Definition

$(\mathcal{H}, \boxplus)$ is a **hyper-semigroup** when

1. $\mathcal{H}$ is a set with a commutative binary operation $\boxplus : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$, which also is associative in the sense that if we define

$$a \boxplus S = S \boxplus a = \bigcup_{s \in S} a \boxplus s, \qquad S_1 \boxplus S_2 := \cup\{s_1 \boxplus s_2 : s_i \in S_i\},$$

then $(a_1 \boxplus a_2) \boxplus a_3 = a_1 \boxplus (a_2 \boxplus a_3)$ for all a_i in $\mathcal{H}$.

Hypergroups, hyperrings, and hyperfields

Definition

1. A **hypergroup** is a hyper-semigroup with a neutral element $0_{\mathcal{H}} \in \mathcal{H}$, called the **hyperzero**, i.e., $0_{\mathcal{H}} \boxplus a = \{a\}$, $\forall a \in \mathcal{H}$, in which every element $a \in \mathcal{H}$ has a unique **hypernegative** $-a \in \mathcal{H}$, in the sense that $0_{\mathcal{H}} \in a \boxplus (-a)$.
   We call $\mathcal{H}$ a **hyperring** when $\mathcal{H}$ also has multiplication distributing over $\boxplus$, $\mathcal{P}^*(\mathcal{H})$ has a natural elementwise multiplication, for which $0_{\mathcal{H}}$ becomes an absorbing element.

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The **hyperpair** of a hyper-semigroup $\mathcal{H}$ is $(\mathcal{A}, \mathcal{A}_0)$, where $\mathcal{A} = \mathcal{P}^*(\mathcal{H})$, and $\mathcal{A}_0 = \{S \in \mathcal{A} : 0 \in S\}$. $\mathcal{A}$ is an nd-semiring under elementwise operations, but surprisingly, in general is not distributive, but only $(\boxplus_i a_i)(\boxplus_j a'_j) \subseteq \boxplus(a_i a'_j)$.

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The supertropical algebra is a hyperpair, where $a \boxplus a := \{c : c \leq a\}$.

Krasner's quotient hyper-semiring construction

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1. Suppose $(\mathcal{S}, +)$ is an additive group and $f : \mathcal{S} \to \bar{\mathcal{S}}$ is any set-theoretic map onto a set $\bar{\mathcal{S}}$. Define the addition $\boxplus : \bar{\mathcal{S}} \to \mathcal{P}^*(\bar{\mathcal{S}})$ by $\bar{a} \boxplus \bar{a'} = \{\overline{a + a'} : f(a) = \bar{a}, f(a') = \bar{a'}\}$, for $a, a' \in \mathcal{S}$. Then $\bar{\mathcal{S}}$ is a hypergroup, where $-\bar{a} = \overline{-a}$.

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2. Suppose that $\mathcal{T}_0$ is a monoid. For any commutative $\mathcal{T}_0$-semiring R, and any subgroup $\mathcal{G}$ of $\mathcal{T}_0$, the set of multiplicative cosets $R/\mathcal{G} = \{b\mathcal{G} : b \in R\}$ has a natural associative hyperaddition given by

$$b_1\mathcal{G} \boxplus b_2\mathcal{G} = \{(a_1 + a_2)\mathcal{G} : \quad a_1 \in b_1\mathcal{G}, \quad a_2 \in b_1\mathcal{G}\}.$$

   Define $\mathcal{A} = \mathcal{P}^*(R/\mathcal{G})$ and $\mathcal{A}_0 = \{S \in \mathcal{A} : 0 \in S\}$. Then $(\mathcal{A}, \mathcal{A}_0)$ is a hyperpair over $\mathcal{T}_0/\mathcal{G}$.

Comments about hypergroups

Hyperfields were exploited by Krasner to prove arithmetic results about fields. Pairs are especially effective with respect to hyperfields, since the polynomials over a hyperfield are not a hyperfield in a natural way, whereas polynomials over a pair are a pair.

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Hyperfields motivate the category theory for pairs as follows:

A function $f : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}_0')$ of pairs is a **weak morphism** if $\sum a_i \in \mathcal{A}_0$ implies $\sum f(a_i) \in \mathcal{A}_0'$.
This matches the definition of weak morphism of hyperfields.

Lie pairs

The **Lie pair** $(\mathcal{L}, \mathcal{L}_0)$ has a Lie bracket satisfying, for all $x, y, z \in \mathcal{L}_0$,

1. $[xx] \in \mathcal{L}_0$,
2. $[xy] + [yx] \in \mathcal{L}_0$,
3. $[[xy]z] + [x[zy]] + [[xz]y] \in \mathcal{L}_0$, called the **Jacobi $\mathcal{L}_0$-identity**.
4. $[z[xy]] + [[zy]x] + [y[xz]] \in \mathcal{L}_0$, the **reflected Jacobi $\mathcal{L}_0$-identity**.
5. If $\sum_i x_i \in \mathcal{L}_0$, then $\sum_i [x_i, y] \in \mathcal{L}_0$, and $\sum_i [y, x_i] \in \mathcal{L}_0$.

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4. $[z[xy]] + [[zy]x] + [y[xz]] \in \mathcal{L}_0$, the **reflected Jacobi $\mathcal{L}_0$-identity**.
5. If $\sum_i x_i \in \mathcal{L}_0$, then $\sum_i [x_i, y] \in \mathcal{L}_0$, and $\sum_i [y, x_i] \in \mathcal{L}_0$.

Ironically the PBW theorem often is easier to prove than in the classical case, since one does not have to worry about negation. But that is the topic of a different talk.

Metatangible pairs

Definition

1. A pair $(\mathcal{A}, \mathcal{A}_0)$ satisfying Property N is **metatangible** if $a_1 + a_2 \in \mathcal{T}_{\mathcal{A}} \cup \mathcal{A}_0$ for all $a_1, a_2 \in \mathcal{T}_{\mathcal{A}}$.

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2. A metatangible pair $(\mathcal{A}, \mathcal{A}_0)$ is **geometric** if $e + e = e$.
3. A metatangible pair $(\mathcal{A}, \mathcal{A}_0)$ is $\mathcal{A}_0$-**bipotent** if $a_1 + a_2 \in \{a_1, a_2\} \cup \mathcal{A}_0$ for all $a_1, a_2 \in \mathcal{T}_{\mathcal{A}}$.

$\mathcal{A}_0$-bipotent pairs of the second kind are geometric, as are many metatangible pairs.

Congruences take the place of ideals in universal algebra, so we deal with congruences, which are defined as equivalence classes which are subalgebras of $\mathcal{A} \times \mathcal{A}$ (in the sense of universal algebra). Joo and Mincheva came up with a brilliant notion for idempotent semirings.

Definition

1. Define the **twist product** given by

$$(a_1, a_2) \cdot_{\mathrm{tw}} (b_1, b_2) = (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1). \qquad (1)$$

   for $a_i \in \mathcal{T}$, $b_i \in \mathcal{A}$.
2. When $\mathcal{A}$ is an nd-semiring, we define the **twist product** on $\mathcal{A} \times \mathcal{A}$ via (1), for all $a_i, b_i \in \mathcal{A}$.
3. The **twist product** $\Omega_1 \cdot_{\mathrm{tw}} \Omega_2 := \{\mathbf{b}_1 \cdot_{\mathrm{tw}} \mathbf{b}_2 : \mathbf{b}_i \in \Omega_i\}$.

The prime spectrum

1. A congruence Ω of $(\mathcal{A}, \mathcal{A}_0)$ is **semiprime** if it satisfies the following condition for a congruence $\Omega_1 \supseteq \Omega$:
   - If $\Omega_1 \cdot_{\mathrm{tw}} \Omega_1 \subseteq \Omega$ then $\Omega_1 = \Omega$.

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   $(\mathcal{A}, \mathcal{A}_0)$ is a **semiprime pair** if the trivial congruence is semiprime.
2. A congruence Ω of $\mathcal{A}$ is **prime** if it satisfies the following condition for congruences Ω_1, $\Omega_2 \supseteq \Omega$:
3. The **prime spectrum** $\mathrm{Spec}(\mathcal{A}, \mathcal{A}_0)$ is the set of prime congruences of $\mathcal{A}$.

1. If $f : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}'_0)$ is a homomorphism of pairs and Ω is a congruence of $(\mathcal{A}, \mathcal{A}_0)$, then $f(\Omega)$ is a congruence of $f(\mathcal{A}, \mathcal{A}_0)$.
2. Any congruence Ω of a pair $(\mathcal{A}, \mathcal{A}_0)$, induces a 1:1 map Ψ from the congruences of $(\mathcal{A}, \mathcal{A}_0)$ containing Ω onto the congruences of $(\mathcal{A}, \mathcal{A}_0)/\Omega$, given by $\Omega' \mapsto \Omega'/\Omega$.
3. For any congruence $\Omega' \supseteq \Omega$, Ψ of (i) induces a homeomorphism from Ω'-$\mathrm{Spec}_{\mathrm{geometric}}(\mathcal{A}, \mathcal{A}_0)$ to (Ω'/Ω)-$\mathrm{Spec}_{\mathrm{geometric}}((\mathcal{A}, \mathcal{A}_0)/\Omega)$.

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The point of all this is that when a semiring pair $(\mathcal{A}, \mathcal{A}_0)$ is $\mathcal{A}_0$-bipotent, then $\mathcal{A}_0 e$ is a bipotent semiring and the lovely Joo-Mincheva theory of idempotent semirings can be lifted.

Theorem

Suppose $(\mathcal{A}, \mathcal{A}_0)$ *is a pair satisfying* $(1 + e + e, e + e) \in \mathrm{Diag}$.

1. *Every semiprime congruence of* $\mathcal{A}$ *contains* $(1, e)$.
2. $\widetilde{\mathrm{Diag}}$-$\mathrm{Spec}(\mathcal{A}, \mathcal{A}_0)$ *is homeomorphic to* $\mathrm{Spec}(\mathcal{A}_0)$
3. *Every maximal chain of prime congruences in* $(\mathcal{A}, \mathcal{A}_0)[\lambda_1, \ldots, \lambda_t]$ *has length* t.

The geometric prime spectrum

We can extend this for pairs.

1. A congruence Ω is **geometrically prime** if $(\overline{\mathcal{A}}, \overline{\mathcal{A}_0}) := (\mathcal{A}, \mathcal{A}_0)/\Omega$ is $\overline{\mathcal{A}_0}$-bipotent.
2. A proper congruence Ω is **geometrically prime-proper** if $(\overline{\mathcal{A}}, \overline{\mathcal{A}_0}) := (\mathcal{A}, \mathcal{A}_0)/\Omega$ is proper $\overline{\mathcal{A}_0}$-bipotent.
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1. Any $\mathcal{A}_0$-bipotent pair of the second kind is proper geometric.

Thus we see that $\mathrm{Spec}_{\mathrm{geometric}}(\mathcal{A}, \mathcal{A}_0)$ lifts $\mathrm{Spec}\,\mathcal{A}_0 e$, and its theory includes the Joo-Mincheva theory.

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For a set $S \subseteq \widehat{A}$, the S-**geometric spectrum** S-$\mathrm{Spec}_{\mathrm{geometric}}(\mathcal{A}, \mathcal{A}_0)$ is the set of geometrically prime congruences of $(\mathcal{A}, \mathcal{A}_0)$ containing S.

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Zariski topology on $\mathrm{Spec}_{\mathrm{geometric}}(\mathcal{A}, \mathcal{A}_0)$ has the closed sets being $\Omega\text{-}\mathrm{Spec}_{\mathrm{geometric}}(\mathcal{A}, \mathcal{A}_0)$, ranging over the congruences Ω of $(\mathcal{A}, \mathcal{A}_0)$.

More recent background

Lorschield (2012) developed an algebraic and geometric theory of "blueprints." In 2016, R introduced general algebraic frameworks called a "triple" $(\mathcal{A}, \mathcal{T}, (-))$, and a "system" $(\mathcal{A}, \mathcal{T}, (-), \preceq)$ to unify various algebraic theories including "classical" algebra, tropical algebra, hyperrings, and fuzzy rings. Baker-Bowler (2019) defined "tracts." J. Jun, K. Mincheva, and R introduced pairs in 2021.

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Balanced elements

The remainder of this file reviews the linear algebra, given in the talk in Kiev.

Definition

Suppose $(\mathcal{A}, \mathcal{A}_0)$ is a pair.

1. An element $b_1 \in \mathcal{A}$ **tangibly balances** $b_2 \in \mathcal{A}$ if there is $a \in \mathcal{T}_0$ such that $b_1 + a \in \mathcal{A}_0$ and $b_2 + a \in \mathcal{A}_0$.
2. The relation ∇ is defined as follows:
    1. For $(\mathcal{A}, \mathcal{A}_0)$ of the first kind, $b_1 \nabla b_2$ if $b_1, b_2 \in \mathcal{A}_0$ or $b_1 + b_2 \in \mathcal{A}_0$.
    2. For $(\mathcal{A}, \mathcal{A}_0)$ of the second kind, $b_1 \nabla b_2$ if b_1 tangibly balances b_2.

Determinants and singularity of matrices

Let us compare the main notions of rank. A **track** of an $n \times n$ matrix $A = (a_{i,j})$ is a product $a_\pi := a_{\pi(1),1} \cdots a_{\pi(n),n}$ for $\pi \in S_n$.
Proceeding further requires the notion of determinant. When $(\mathcal{A}, \mathcal{A}_0)$ is of the first kind, we *always* take the permanent, in defining $|A| := \sum_\pi a_\pi$. We say that a square matrix A is **singular** if $|A| \in \mathcal{A}_0$; otherwise A is **nonsingular**.

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In general we need a more intricate approach:

$$|A|_+ = \sum_{\mathrm{sgn}(\pi)\ \mathrm{even}} a_\pi, \qquad |A|_- = \sum_{\mathrm{sgn}(\pi)\ \mathrm{odd}} a_\pi. \tag{2}$$

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$$|A|_+ = \sum_{\mathrm{sgn}(\pi) \text{ even}} a_\pi, \qquad |A|_- = \sum_{\mathrm{sgn}(\pi) \text{ odd}} a_\pi. \tag{2}$$

A square matrix A is singular if $|A|_+ \nabla |A|_-$; otherwise A is nonsingular. The **submatrix rank** of A is the largest size of a nonsingular square submatrix of A.

This procedure, called **doubling**, can be viewed more generally to circumvent negation which one might expect in various situations, such as in super-semialgebras

Vector space pairs and dependence

Definition

Fixing n, take $\mathcal{V} := \mathcal{A}^{(n)}$, which has the *standard base* $\{e_i = (0, \dots, 0, 1, 0, \dots, 0) : 1 \leq i \leq n\}$; we define $\mathcal{T}_{\mathcal{V}} = \cup_{i=1}^{n} \mathcal{T} e_i$.
A **vector space pair** over a semiring pair $(\mathcal{A}, \mathcal{A}_0)$ is $(\mathcal{V}, \mathcal{V}_0)$, where $\mathcal{V}_0 = \mathcal{A}_0^{(n)}$. The $\mathcal{T}$-module operations are defined componentwise.

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Definition

A set of vectors $\{\mathbf{v}_i \in \mathcal{V} : i \in I\}$ is V_0-**dependent** (written **dependent** for short), if $\sum_{i \in I'} a_i \mathbf{v}_i \in V_0$ for some nonempty finite subset $I' \subseteq I$ and $a_i \in \mathcal{T}$.

The **row rank** of a matrix is the maximal number of $\mathcal{A}_0$-independent rows.

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The **column rank** of a matrix is the maximal number of $\mathcal{A}_0$-independent columns.

The rank conditions

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2. **Condition** A2: The submatrix rank is greater than or equal to the row rank and the column rank.
3. **Condition A2′**: The rows of an $m \times n$ matrix are dependent if $m < n$.

Moduli

A **modulus** on an admissible semiring $\mathcal{A}$ with **values** in a bipotent semiring $\mathcal{G}$ is a semiring homomorphism $\mu : \mathcal{A} \to \mathcal{G}$. Condition A1 holds in the presence of a modulus and in fact one sometimes can get a version of Cramer's rule.

Condition A2 is subtler. Condition A2 holds for square matrices, in certain special cases in Theorem P, and over pairs of "tropical type."
On the other hand, a basic counterexample to Condition A2 of the second kind (which goes back to work of Gaubert and his colleagues) is the idempotent "sign" semiring pair $(\mathcal{A}, \mathcal{A}_0) = (\{1, 0, -1, \infty\}, \{\infty\})$, with $1 + (-1) = \infty$, $+$ for $+1$ and $-$ for -1.

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$$\begin{pmatrix} + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix} \tag{3}$$

has row rank 3. On the other hand, each 3×3 minor is singular.

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Hence there is a 4×4 counterexample. Ironically, there is no 3×3 counterexample. Partial positive results exist for nonsquare matrices.