Linear algebra over semiring pairs

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a joint work with

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Abstract

This is the continuation of an ongoing project to find a general algebraic framework for semiring theory. The structure theory of semirings is quite challenging, largely because of the lack of negation, and such basic properties such as unique factorization of polynomials, multiplicativity of determinants, and the characteristic polynomial of a matrix, all fail. (In fact in the max-plus algebra, the sum of two nonzero elements is *never* zero!) Consequently 0 is replaced by a distinguished ideal $\mathcal{A}_0$ of $\mathcal{A}$, and $(\mathcal{A}, \mathcal{A}_0)$ is called a *pair*.

In this talk we discuss results obtained in the last two years on linear algebra over a (not necessarily distributive) semiring pair, with a range of applications to tropical algebra as well as related areas such as hyperrings and fuzzy rings. First we present pairs with their morphisms, called "weak morphisms." We pay special attention to supertropical pairs and hyperpairs.

Then we turn to matrices and the question of whether the row rank, column rank, and submatrix rank of a matrix are equal. The submatrix rank is less than or equal to the row rank and the column rank in many cases, including "metatangible pairs" with unique negation, but there is a counterexample to equality, discovered some time ago by the second author, which we discussed in the talk two years ago. There are situations when equality holds, encompassing results by Akian, Gaubert, Guterman, Izhakian, Knebusch, and Rowen, including versions of Cramer's rule. We pay special attention to a recent question of Baker and Zhang whether $n + 1$ vectors of length n need be dependent.

Keywords

Cramer's rule, hypergroup, hyperfield, Jacobi's algorithm, linear algebra, submatrix rank, metatangible, negation map, pair, Property N, semifield, semiring, supertropical algebra, tropical

References

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