Rings Closed to Semiregular

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joint work with

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\[ R := \text{a ring with identity} \]
\[ J := \text{the Jacobson radical of } R \]
Definitions

$R :=$ a ring with identity

$J :=$ the Jacobson radical of $R$

**Definition**

An element $a \in R$ is called **regular**
Definitions

\( R \) := a ring with identity

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### Definition

An element \( a \in R \) is called **regular** if there exists \( b \in R \) such that \( a = aba \).
Definitions

\(R := a \text{ ring with identity}\)

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**Definition**

An element \(a \in R\) is called **regular** if there exists \(b \in R\) such that \(a = aba\).

\(R\) is called **regular** if every element of \(R\) is regular.

(von Neumann, 1936).
Definitions

$R := a$ ring with identity

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Definition

An element $a \in R$ is called \textit{regular} if there exists $b \in R$ such that $a = aba$.

$R$ is called \textit{regular} if every element of $R$ is regular. (von Neumann, 1936).

These rings were introduced by von Neumann as co-ordinate rings of infinite dimensional projective and continuous geometry.
Rings closed to regular:

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
strongly regular
strongly regular $\Rightarrow$ unit–regular
Rings closed to regular:

- strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
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Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular
Rings closed to regular:

strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular
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- strongly regular
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- strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

- strongly regular $\Rightarrow$ strongly $\pi$–regular
Rings closed to regular:

strongly regular $\implies$ unit–regular $\implies$ one–sided unit–regular $\implies$
regular $\implies$ weakly regular

strongly regular $\implies$ strongly $\pi$–regular $\implies$ unit $\pi$–regular
Rings closed to regular:

- strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

- strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular
Rings closed to regular:

- strongly regular \Rightarrow unit-regular \Rightarrow one-sided unit-regular \Rightarrow regular \Rightarrow weakly regular

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Definition 1.1

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Idempotents lift modulo $J$ if whenever $a^2 - a \in J$ for any $a \in R$, there exists $e^2 = e \in R$ such that $a - e \in J$. 
Semiregular ring:

Theorem 1.2

TFAE for a ring $R$:
1. $R/J$ is regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exists a regular element $d$ in $R$ such that $a - d \in J$.
3. For any $a \in R$, there exists a regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
4. For any $a \in R$, there exists an idempotent $e \in aR$ such that $(1 - e)a \in J$.
5. For any $a \in R$, there exists an idempotent $e \in Ra$ such that $a(1 - e) \in J$.

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strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
regular $\Rightarrow$ weakly regular
Definition 2.1

An element of $a$ of $R$ is called (one–sided) unit–regular

\[ a = auau \]

$R$ is called (one–sided) unit–regular if every element of $R$ is (one–sided) unit regular (Ehrlich, 1968).

Definition 2.2

$R$ is said to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is (right) invertible (Bass, 1964).

If $R$ is regular, then $R$ has stable range 1 $\iff$ $R$ is unit–regular (Goodearl, 1979).

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$R$ is said to have **stable range 1** if for any $a, b \in R$ satisfying $aR + bR = R$, 

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If \( R \) is regular, then
\( R \) has stable range 1 $\iff$ \( R \) is unit–regular (Goodearl, 1979).
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Semi Unit–Regular Rings:

**Theorem 2.3**
Rings Closed to Semiregular

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Semi Unit–Regular Rings:

Theorem 2.3

TFAE for a ring $R$:

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Semi Unit–Regular Rings:

**Theorem 2.3**

TFAE for a ring \( R \):

1. \( R/J \) is unit regular and idempotents lift modulo \( J \).
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strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

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4. For any \( a \in R \), there exists an idempotent \( e \) and a unit \( b \) in \( R \) such that \( e \in aR \), \( (1 - e)a \in J \), \( ba - (ba)^2 \in J \).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

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4. For any $a \in R$, there exists an idempotent $e$ and a unit $b$ in $R$ such that $e \in aR$, $(1 - e)a \in J$, $ba - (ba)^2 \in J$.

5. For any $a \in R$, there exists an idempotent $e$ and a unit $b$ in $R$ such that $e \in Ra$, $a(1 - e) \in J$, $ab - (ab)^2 \in J$. 
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Semi Unit–Regular Rings:

**Theorem 2.3**

TFAE for a ring $R$:

1. $R/J$ is unit regular and idempotents lift modulo $J$

2. For any $a \in R$, there exists a unit–regular element $d$ in $R$ such that $a - d \in J$.

3. For any $a \in R$, there exists a unit–regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.

4. For any $a \in R$, there exists an idempotent $e$ and a unit $b$ in $R$ such that $e \in aR$, $(1 - e)a \in J$, $ba - (ba)^2 \in J$.

5. For any $a \in R$, there exists an idempotent $e$ and a unit $b$ in $R$ such that $e \in Ra$, $a(1 - e) \in J$, $ab - (ab)^2 \in J$.

6. $R$ is a semiregular ring with stable range 1.
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Definition 2.4
strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

Definition 2.4
A ring $R$ is said to have **weak stable range 1**
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

**Definition 2.4**

A ring $R$ is said to have **weak stable range 1** if for any $a, b \in R$ satisfying $aR + bR = R$,
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
regular $\Rightarrow$ weakly regular

Definition 2.4

A ring $R$ is said to have weak stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists an element $y$ in $R$ such that $a + by$ is a one-sided unit (Wu, 1994).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

**Definition 2.4**

A ring $R$ is said to have **weak stable range 1** if for any $a, b \in R$ satisfying $aR + bR = R$, there exists an element $y$ in $R$ such that $a + by$ is a one-sided unit (Wu, 1994).

If $R$ is regular, then $R$ has weak stable range 1 $\Leftrightarrow$ $R$ is one–sided unit–regular.
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Semi One–sided Unit–Regular Rings:

Theorem 2.5
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Semi One–sided Unit–Regular Rings:

**Theorem 2.5**

TFAE for a ring $R$:

1. $R/J$ is one-sided unit–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exists a one-sided unit regular element $d \in aR$ (resp. $d \in aRa$) such that $a - d \in J$.
3. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in aRb$ such that $(1 - e)a \in J$ and $ba - (ba)^2 \in J$.
4. For any $a \in R$, there exist a one-sided unit $b \in R$ and an idempotent $e \in bRa$ such that $a(1 - e) \in J$ and $ab - (ab)^2 \in J$.

5. $R$ is a semiregular ring with weak stable range 1.
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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Semi One–sided Unit–Regular Rings:

**Theorem 2.5**

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Strongly regular \( \Rightarrow \) unit–regular \( \Rightarrow \) one–sided unit–regular \( \Rightarrow \) regular \( \Rightarrow \) weakly regular

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strongly regular \Rightarrow \text{unit–regular} \Rightarrow \text{one–sided unit–regular} \Rightarrow \text{regular} \Rightarrow \text{weakly regular}

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Semi One–sided Unit–Regular Rings:

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5. \( R \) is a semiregular ring with weak stable range 1.
strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

$R$ is semi unit–regular ⇔

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$R$ is semi unit–regular $\iff$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_i Re_j$ are semi unit–regular (H. Chen, M. Chen, 2003).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Theorem 2.6

$R$ is semi one–sided unit–regular $\iff$
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
regular $\Rightarrow$ weakly regular

Theorem 2.6

$R$ is semi one–sided unit–regular $\iff$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_iRe_j$ are semi one–sided unit–regular.

Sketch of the proof
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

**Theorem 2.6**

$R$ is semi one–sided unit–regular $\iff$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_iRe_j$ are semi one–sided unit–regular.

**Sketch of the proof**

$(\Leftarrow)$: $\Diamond$ $R$ is semiregular (Chen-Chen, 2003)
Theorem 2.6

*R* is semi one–sided unit–regular ⇔ there exists a complete orthogonal set \( \{e_1, \ldots, e_n\} \) of idempotents of *R* such that \( e_iRe_j \) are semi one–sided unit–regular.

Sketch of the proof

(⇐): ♦ *R* is semiregular (Chen-Chen, 2003)
♦ Aim: \( \text{End}(\oplus_{i=1}^n e_iR) \) has weak stable range 1.
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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**Theorem 2.6**

$R$ is semi one–sided unit–regular $\iff$ there exists a complete
orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that
$e_i Re_j$ are semi one–sided unit–regular.

**Sketch of the proof**

$(\Leftarrow)$: ◇ $R$ is semiregular (Chen-Chen, 2003)
◇ Aim: $\text{End}(\bigoplus_{i=1}^{n} e_i R)$ has weak stable range 1.
◇ $\text{End}(e_i R) \cong e_i Re_i$ is semi one–sided unit–regular, $\forall i$. 

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Theorem 2.6

$R$ is semi one–sided unit–regular $\iff$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_iRe_j$ are semi one–sided unit–regular.

Sketch of the proof

$(\Leftarrow)$: ◊ $R$ is semiregular (Chen-Chen, 2003)
◊ Aim: $\text{End}(\bigoplus_{i=1}^n e_iR)$ has weak stable range 1.
◊ $\text{End}(e_iR) \cong e_iRe_i$ is semi one–sided unit–regular, $\forall i$.
◊ All $e_iR$ have the fep (Warfield, 1972).
strongly regular \Rightarrow unit–regular \Rightarrow \textbf{one–sided unit–regular} \Rightarrow regular \Rightarrow weakly regular

**Theorem 2.6**

$R$ is semi one–sided unit–regular $\iff$ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_iRe_j$ are semi one–sided unit–regular.

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($\Leftarrow$): ◆ $R$ is semiregular (Chen-Chen, 2003)

◆ Aim: End($\bigoplus_{i=1}^{n} e_i R$) has weak stable range 1.

◆ End$(e_i R) \cong e_i R e_i$ is semi one–sided unit–regular, $\forall i$.

◆ All $e_i R$ have the fep (Warfield, 1972).

◆ All $e_i R$ satisfy outer weak cancellation (Li-Tong, 2002).
strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

Theorem 2.6

$R$ is semi one–sided unit–regular ⇔ there exists a complete orthogonal set $\{e_1, \ldots, e_n\}$ of idempotents of $R$ such that $e_i Re_j$ are semi one–sided unit–regular.

Sketch of the proof

(⇐): ◇ $R$ is semiregular (Chen-Chen, 2003)
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$M$ satisfies **outer weak cancellation** if $M \oplus K \cong M \oplus L$, then there exist a splitting epimorphisms between $K$ and $L$ (Wu, 1996).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

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strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

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strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

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◊ $\text{End}(\bigoplus_{i=1}^{n} e_iR)$ has weak stable range 1 (Li-Tong, 2002).
◊ $R$ is semiregular with weak stable range 1.
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

**Definition 3.1**

An element $a$ in $R$ is called **strongly regular**
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
regular $\Rightarrow$ weakly regular

**Definition 3.1**

An element $a$ in $R$ is called **strongly regular** if there exists $x \in R$ such that $a = a^2 x$. 
strongly regular ⇒ unit–regular ⇒ one–sided unit–regular ⇒ regular ⇒ weakly regular

Definition 3.1

An element $a$ in $R$ is called strongly regular if there exists $x \in R$ such that $a = a^2 x$.

If any element in $R$ is strongly regular,
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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**Definition 3.1**

An element $a$ in $R$ is called **strongly regular** if there exists
$x \in R$ such that $a = a^2 x$.

If any element in $R$ is strongly regular, then $R$ is called a
**strongly regular ring** (Arens-Kaplansky, 1948).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

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Well-known characterization of strongly regular rings:
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Well-known characterization of strongly regular rings:

$R$ is strongly regular

$\iff$

\[ R \text{ is strongly regular } \iff R \text{ is regular and abelian (i.e. every idempotent is central)} \iff R \text{ is unit–regular and abelian} \iff \text{For any } a \in R, \text{ there exist an idempotent } e \text{ and a unit } u \text{ in } R \text{ such that } a = eu \text{ and } eu = ue (\text{Goodearl, 1979}) \]
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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Rings Closed to Semiregular

- 1. Semiregular Rings
- 2. Semi (one–sided) unit–regular
- 3. Semi strongly regular
- 4. Semi \(\pi\)–regular
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For any $a \in R$, there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = eu$ and $eu = ue$

(Goodearl, 1979)
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$

regular $\Rightarrow$ weakly regular

Semi strongly regular ring:

Theorem 3.2
strongly regular \Rightarrow unit-regular \Rightarrow one-sided unit-regular \Rightarrow regular \Rightarrow weakly regular

Semi strongly regular ring:

**Theorem 3.2**

TFAE for a ring $R$:

1. $R/J$ is strongly regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a unit $u$ and an idempotent $e$ in $R$ (resp. in $aR$) such that $a - eu \in J$ and $eu - ue \in J$.
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4. For any $a \in R$, there exists a unit $u$ and an idempotent $e \in Ra$ such that $a(1 - e) \in J$ and $au = ua$ is an idempotent in $R/J$. 
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

Semi strongly regular ring:

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strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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4. For any $a \in R$, there exists a unit $u$ and an idempotent $e \in Ra$ such that $a(1 - e) \in J$ and $\overline{au} = \overline{ua}$ is an idempotent in $R/J$. 
**strongly regular** \(\Rightarrow\) **unit–regular** \(\Rightarrow\) **one–sided unit–regular** \(\Rightarrow\) **regular** \(\Rightarrow\) **weakly regular**

**Definition 3.3**

A ring \(R\) is said to have **unit stable range 1**
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

**Definition 3.3**

A ring $R$ is said to have **unit stable range 1** if, for any $a, b \in R$, satisfying $aR + bR = R$, there exists a unit $u \in R$ such that $a + bu$ is a unit. (Goodearl-Menal, 1988).
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

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So we have the following implications:
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
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unit stable range 1 $\Rightarrow$ stable range 1 $\Rightarrow$ weak stable range 1
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So we have the following implications:

- Unit stable range 1 $\Rightarrow$ stable range 1 $\Rightarrow$ weak stable range 1
- Semi strongly regular $\Rightarrow$ semi unit–regular $\Rightarrow$ semi one–sided unit–regular
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$
regular $\Rightarrow$ weakly regular

semi strongly regular $\nLeftrightarrow$ semiregular and unit stable range 1
strongly regular \Rightarrow \text{unit–regular} \Rightarrow \text{one–sided unit–regular} \Rightarrow \text{regular} \Rightarrow \text{weakly regular}

semi strongly regular \not\iff \text{semiregular and unit stable range 1}

Example 3.4

Example 3.5

R = \mathbb{Z}_2 \ast \mathbb{Z}_2

R = \text{regular}

R = \text{non-central idempotents}
strongly regular $\Rightarrow$ unit–regular $\Rightarrow$ one–sided unit–regular $\Rightarrow$ regular $\Rightarrow$ weakly regular

semi strongly regular $\nleftrightarrow$ semiregular and unit stable range 1

Example 3.4

$R = \mathbb{Z}_2$

$\Diamond$ is semi strongly regular
Rings Closed to Semiregular

AYDOĞDU, ÖZCAN

1. Semiregular Rings
2. Semi (one-sided) unit-regular
3. Semi strongly regular $\nLeftarrow$ semiregular and unit stable range 1
4. Semi $\pi$-regular
5. Semi unit $\pi$-regular
6. Semi strongly $\pi$-regular
7. Semi right weakly $\pi$-regular
8. General

**Strongly regular** $\Rightarrow$ **unit-regular** $\Rightarrow$ **one-sided unit-regular** $\Rightarrow$ **regular** $\Rightarrow$ **weakly regular**

**Example 3.4**

$R = \mathbb{Z}_2$

- is semi strongly regular
- does not have unit stable range 1

**Example 3.5**

$R = M_2(\mathbb{Z}_2)$

- is semiregular
- has unit stable range 1 (Chen, 2000)
- is not semi strongly regular since it has non-central idempotents
**strongly regular** ⇒ **unit–regular** ⇒ **one–sided unit–regular** ⇒ **regular** ⇒ **weakly regular**

**semi strongly regular** ⊖ semiregular and unit stable range 1

**Example 3.4**

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semi strongly regular \(\not\Leftrightarrow\) semiregular and unit stable range 1

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An element \(a\) of \(R\) is called \(\pi\)-regular.
strongly regular ⇒ strongly \( \pi \)-regular ⇒ unit \( \pi \)-regular ⇒ \( \pi \)-regular ⇒ weakly \( \pi \)-regular

**Definition 4.1**
An element \( a \) of \( R \) is called \( \pi \)-regular if a power of \( a \) is regular.
Rings Closed to Semiregular

AYDOĞDU, ÖZCAN

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi π–regular
5. Semi unit π–regular
6. Semi strongly π–regular
7. Semi right weakly π–regular
8. General

Definition 4.1

An element $a$ of $R$ is called $\pi$–regular if a power of $a$ is regular. If every element of $R$ is $\pi$–regular, then $R$ is called $\pi$-regular (Kaplansky, 1951).

strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular
Rings Closed to Semiregular

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An element \( a \) of \( R \) is called \( \pi \)–regular if a power of \( a \) is regular.

If every element of \( R \) is \( \pi \)–regular, then \( R \) is called \( \pi \)-regular (Kaplansky, 1951).

\( R \) is \( \pi \)–regular \( \iff \)
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

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An element $a$ of $R$ is called $\pi$–regular if a power of $a$ is regular. If every element of $R$ is $\pi$–regular, then $R$ is called $\pi$–regular (Kaplansky, 1951).

$R$ is $\pi$–regular $\iff$ for any $a \in R$, there exist a positive integer $n$ and a $\pi$–regular element $d \in R$ such that $a^n - d = 0$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi $\pi$–regular ring:

G. Xiao, W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005
strongly regular \Rightarrow strongly \ \pi-regular \Rightarrow unit \ \pi-regular \Rightarrow \\
\pi-regular \Rightarrow weakly \ \pi-regular

Semi \ \pi-regular \ ring:

G. Xiao, W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

$R/J$ is $\pi$–regular and idempotents lift modulo $J$. 

$\iff$
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular ⇒ weakly $\pi$–regular

Semi $\pi$–regular ring:

G. Xiao, W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

$R/J$ is $\pi$–regular and idempotents lift modulo $J$.

$\iff$

For any $a \in R$, there exist a positive integer $n$ and a regular element $d \in R$ such that $a^n - d \in J$.

$\iff$
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ 
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G. Xiao, W. Tong; *Generalizations of semiregular rings*, Comm. Algebra, 2005

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$\iff$
For any $a \in R$, there exist a positive integer $n$ and a regular element $d \in R$ such that $a^n - d \in J$.
$\iff$
For any $a \in R$, there exist a positive integer $n$ and an idempotent $e \in a^nR$ such that $(1 - e)a^n \in J$.
$\iff$
Strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

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$\iff$

For any $a \in R$, there exist a positive integer $n$ and an idempotent $e \in Ra^n$ such that $a^n(1 - e) \in J$. 
strongly regular \Rightarrow strongly \pi-regular \Rightarrow unit \pi-regular \Rightarrow 
\pi-regular \Rightarrow weakly \pi-regular

Semi \pi-regular ring:

<table>
<thead>
<tr>
<th>Theorem 4.2</th>
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1. $R/J$ is $\pi$–regular and idempotents lift modulo $J$.
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3. For any $a \in R$, there exist a positive integer $n$ and a $\pi$–regular element $d \in a^nR$ (resp. $d \in a^nRa^n$) such that $a^n - d \in J$. |
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AYDOĞDU, ÖZCAN

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TFAE for a ring $R$:
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strongly regular ⇒ strongly π–regular ⇒ unit π–regular ⇒ π–regular ⇒ weakly π–regular

Semi π–regular ring:

Theorem 4.2
TFAE for a ring $R$:

- $R/J$ is π–regular and idempotents lift modulo $J$. 

- For any $a \in R$, there exist a positive integer $n$ and a π–regular element $d$ of $R$ such that $a^n - d \in J$.

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strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

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**Semi $\pi$–regular ring:**

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1. $R/J$ is $\pi$–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a positive integer $n$ and a $\pi$–regular element $d$ of $R$ such that $a^n - d \in J$. 
strongly regular ⇒ strongly π–regular ⇒ unit π–regular ⇒ π–regular ⇒ weakly π–regular

Semi π–regular ring:

**Theorem 4.2**

**TFAE for a ring \( R \):**

1. \( R/J \) is π–regular and idempotents lift modulo \( J \).
2. For any \( a \in R \), there exist a positive integer \( n \) and a π–regular element \( d \) of \( R \) such that \( a^n - d \in J \).
3. For any \( a \in R \), there exist a positive integer \( n \) and a π–regular element \( d \in a^n R \) (resp. \( d \in a^n Ra^n \)) such that \( a^n - d \in J \).
Definition 5.1

An element $a$ of $R$ is called unit $\pi$–regular.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Definition 5.1
An element $a$ of $R$ is called unit $\pi$–regular if there exists a positive integer $n$ such that $a^n$ is unit–regular.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 5.1**

An element $a$ of $R$ is called **unit $\pi$–regular** if there exists a positive integer $n$ such that $a^n$ is unit–regular. $R$ is called **unit $\pi$–regular** if every element of $R$ is unit $\pi$–regular (Chen, 1998).
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi unit $\pi$–regular ring:

Theorem 5.2
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi unit $\pi$–regular ring:

**Theorem 5.2**

TFAE for a ring $R$:

1. $R/J$ is unit $\pi$–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a positive integer $n$ and a unit ($\pi$-regular) element $d \in R$ (resp. $d \in a^n R$) such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^n R$, $(1 - e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$.
4. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in Ra^n$, $a^n(1 - e) \in J$ and $a^n b - (a^n b)^2 \in J$. 

Rings Closed to Semiregular

AYDOĞDU, ÖZCAN

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
strongly regular \implies \text{strongly } \pi-\text{regular} \implies \text{unit } \pi-\text{regular} \implies 
\pi-\text{regular} \implies \text{weakly } \pi-\text{regular} 

Semi unit \pi-\text{regular} ring:

**Theorem 5.2**

TFAE for a ring $R$:

1. $R/J$ is unit $\pi$-regular and idempotents lift modulo $J$. 
2. For any $a \in R$, there exist a positive integer $n$ and a unit (\pi-regular element $d \in R$ (resp. $d \in a^nR$) such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^nR$, $(1-e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$. 
4. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in Ra^n$, $a^n(1-e) \in J$ and $a^n b - (a^n b)^2 \in J$. 
5. Semi unit $\pi$-regular 
6. Semi strongly $\pi$-regular 
7. Semi right weakly $\pi$-regular 
8. General
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi unit $\pi$–regular ring:

**Theorem 5.2**

TFAE for a ring $R$:

1. $R/J$ is unit $\pi$–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a positive integer $n$ and a unit ($\pi$–)regular element $d \in R$ (resp. $d \in a^n R$) such that $a^n - d \in J$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi unit $\pi$–regular ring:

**Theorem 5.2**

**TFAE for a ring $R$:**

1. $R/J$ is unit $\pi$–regular and idempotents lift modulo $J$.

2. For any $a \in R$, there exist a positive integer $n$ and a unit (\(\pi\)-)regular element $d \in R$ (resp. $d \in a^n R$) such that $a^n - d \in J$.

3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^n R$, $(1 - e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$. 
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular ⇒ weakly $\pi$–regular

Semi unit $\pi$–regular ring:

Theorem 5.2

TFAE for a ring $R$:

1. $R/J$ is unit $\pi$–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a positive integer $n$ and a unit ($\pi$-)regular element $d \in R$ (resp. $d \in a^nR$) such that $a^n - d \in J$.
3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in a^nR$, $(1 - e)a^n \in J$ and $ba^n - (ba^n)^2 \in J$.
4. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ and a unit $b \in R$ such that $e \in Ra^n$, $a^n(1 - e) \in J$ and $a^nb - (a^nb)^2 \in J$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called **strongly $\pi$–regular**
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called **strongly $\pi$–regular** if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Definition 6.1

An element $a$ of $R$ is called strongly $\pi$–regular if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called strongly $\pi$–regular if every $a \in R$ is a strongly $\pi$–regular element, equivalently
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called strongly $\pi$–regular if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called strongly $\pi$–regular if every $a \in R$ is a strongly $\pi$–regular element, equivalently $R$ has DCC on the set $\{a^nR\}$ for any $a \in R$ (Kaplansky, 1950).
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called strongly $\pi$–regular if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called strongly $\pi$–regular if every $a \in R$ is a strongly $\pi$–regular element, equivalently $R$ has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly $\pi$–regular rings are $\pi$–regular (Azumaya, 1954).
strongly regular \Rightarrow \textit{strongly } \pi\text{-regular} \Rightarrow \textit{unit } \pi\text{-regular} \Rightarrow \pi\text{-regular} \Rightarrow \textit{weakly } \pi\text{-regular}

Definition 6.1

An element $a$ of $R$ is called strongly $\pi\text{-regular}$ if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called strongly $\pi\text{-regular}$ if every $a \in R$ is a strongly $\pi\text{-regular}$ element, equivalently $R$ has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly $\pi\text{-regular}$ rings are $\pi\text{-regular}$ (Azumaya, 1954).

Strongly $\pi\text{-regular}$ rings have stable range 1 (Ara, 1996).
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒
$\pi$–regular ⇒ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called **strongly $\pi$–regular** if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called **strongly $\pi$–regular** if every $a \in R$ is a strongly $\pi$–regular element, equivalently $R$ has DCC on the set $\{a^n R\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly $\pi$–regular rings are $\pi$–regular (Azumaya, 1954).

Strongly $\pi$–regular rings have stable range 1 (Ara, 1996).

Strongly $\pi$–regular rings are unit $\pi$–regular.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called strongly $\pi$–regular if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = xa^{n+1}$.

$R$ is called strongly $\pi$–regular if every $a \in R$ is a strongly $\pi$–regular element, equivalently $R$ has DCC on the set $\{a^nR\}$ for any $a \in R$ (Kaplansky, 1950).

Strongly $\pi$–regular rings are $\pi$–regular (Azumaya, 1954). Strongly $\pi$–regular rings have stable range 1 (Ara, 1996). Strongly $\pi$–regular rings are unit $\pi$–regular.

$R$ is strongly $\pi$–regular ring $\iff$
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 6.1**

An element $a$ of $R$ is called **strongly $\pi$–regular** if there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$ and $a^n = x a^{n+1}$.

$R$ is called **strongly $\pi$–regular** if every $a \in R$ is a strongly $\pi$–regular element, equivalently $R$ has DCC on the set \( \{a^n R\} \) for any $a \in R$ (Kaplansky, 1950).

Strongly $\pi$–regular rings are $\pi$–regular (Azumaya, 1954). Strongly $\pi$–regular rings have stable range 1 (Ara, 1996). Strongly $\pi$–regular rings are unit $\pi$–regular.

$R$ is strongly $\pi$–regular ring $\iff$ for any $a \in R$, there exists a positive integer $n$ such that $a^n = eu = ue$ for some idempotent $e \in R$ and some unit $u \in R$ (Chin, 2004).
Rings Closed to Semiregular

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1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi \( \pi \)–regular
5. Semi unit \( \pi \)–regular
6. Semi strongly \( \pi \)–regular
7. Semi right weakly \( \pi \)–regular
8. General

strongly regular \( \Rightarrow \) strongly \( \pi \)–regular \( \Rightarrow \) unit \( \pi \)–regular \( \Rightarrow \) \( \pi \)–regular \( \Rightarrow \) weakly \( \pi \)–regular

Semi strongly \( \pi \)–regular ring

Theorem 6.2
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi strongly $\pi$–regular ring

**Theorem 6.2**

TFAE for a ring $R$:

1. $R/J$ is strongly $\pi$–regular and idempotents lift modulo $J$.
2. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ (resp. $e \in a^n R$) and a unit $u \in R$ such that $a^n e - eu \in J$ and $eu - ue \in J$.
3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in a^n R$ and a unit $u \in R$ such that $(1 - e)a^n \in J$ and $a^n u = u a^n$ is an idempotent in $R/J$.
4. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in Ra^n$ and a unit $u \in R$ such that $a^n(1 - e) \in J$ and $a^n u = u a^n$ is an idempotent in $R/J$.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi strongly $\pi$–regular ring

**Theorem 6.2**

TFAE for a ring $R$:

1. $R/J$ is strongly $\pi$–regular and idempotents lift modulo $J$.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi strongly $\pi$–regular ring

**Theorem 6.2**

TFAE for a ring $R$:

1. $R/J$ is strongly $\pi$–regular and idempotents lift modulo $J$.

2. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ (resp. $e \in a^nR$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi strongly $\pi$–regular ring

Theorem 6.2

TFAE for a ring $R$:

1. $R/J$ is strongly $\pi$–regular and idempotents lift modulo $J$.

2. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ (resp. $e \in a^n R$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$.

3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in a^n R$ and a unit $u \in R$ such that $(1 - e)a^n \in J$ and $a^n u = u a^n$ is an idempotent in $R/J$.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Semi strongly $\pi$–regular ring

**Theorem 6.2**

TFAE for a ring $R$:

1. $R/J$ is strongly $\pi$–regular and idempotents lift modulo $J$.

2. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in R$ (resp. $e \in a^n R$) and a unit $u \in R$ such that $a^n - eu \in J$ and $eu - ue \in J$.

3. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in a^n R$ and a unit $u \in R$ such that $(1 - e)a^n \in J$ and $a^n u = ua^n$ is an idempotent in $R/J$.

4. For any $a \in R$, there exist a positive integer $n$, an idempotent $e \in Ra^n$ and a unit $u \in R$ such that $a^n(1 - e) \in J$ and $a^n u = ua^n$ is an idempotent in $R/J$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Theorem 6.3

If, for any $a \in R$, there exist a positive integer $n$ and strongly $\pi$–regular element $d$ such that $a^n - d \in J$, then $R$ is semi strongly $\pi$–regular ring.

The converse of Theorem 6.3 is an open question.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Theorem 6.3
If, for any $a \in R$, there exist a positive integer $n$ and strongly $\pi$–regular element $d$ such that $a^n - d \in J$, 

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5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
Theorem 6.3

If, for any \( a \in R \), there exist a positive integer \( n \) and strongly \( \pi \)-regular element \( d \) such that \( a^n - d \in J \), then \( R \) is semi strongly \( \pi \)-regular ring.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Theorem 6.3

If, for any $a \in R$, there exist a positive integer $n$ and strongly $\pi$–regular element $d$ such that $a^n - d \in J$, then $R$ is semi strongly $\pi$–regular ring.

The converse of Theorem 6.3 is an open question.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.1**

An element $a \in R$ is called **right weakly $\pi$–regular**.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.1**

An element $a \in R$ is called right weakly $\pi$–regular if there exists a positive integer $n$ such that $a^n R = (a^n R)^2$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.1**

An element $a \in R$ is called **right weakly $\pi$–regular** if there exists a positive integer $n$ such that $a^n R = (a^n R)^2$.

$R$ is called **right weakly $\pi$–regular** if any element of $R$ is right weakly $\pi$–regular (Gupta, 1977).
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular ⇒ weakly $\pi$–regular

**Definition 7.1**

An element $a \in R$ is called right weakly $\pi$–regular if there exists a positive integer $n$ such that $a^n R = (a^n R)^2$. $R$ is called right weakly $\pi$–regular if any element of $R$ is right weakly $\pi$–regular (Gupta, 1977).

$R$ is right weakly $\pi$–regular

$\iff$
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.1**

An element $a \in R$ is called **right weakly $\pi$–regular** if there exists a positive integer $n$ such that $a^n R = (a^n R)^2$.

$R$ is called **right weakly $\pi$–regular** if any element of $R$ is right weakly $\pi$–regular (Gupta, 1977).

$R$ is right weakly $\pi$–regular

$\iff$

for any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d = 0$. 
Rings Closed to Semiregular

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6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General

strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

(∗) For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$. 

Any local ring satisfies (∗).

(∗) $\not\Rightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$.

⋄ $R$ is local.

⋄ $R$ satisfies (∗).

⋄ $R$ is not right weakly $\pi$–regular because $x^n/x \not\in x^nRx$ for all positive integer $n$. 

For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$. 

Any local ring satisfies (∗).

(∗) $\not\Rightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$.

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strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

(∗) For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.

• Any local ring satisfies (∗).

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6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular ⇒ weakly $\pi$–regular

\((\ast)\) For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies \((\ast)\).
- \((\ast) \not\Rightarrow\) right weakly $\pi$–regular:
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

\[ (*) \text{ For any } a \in R, \text{ there exist a positive integer } n \text{ and a right weakly } \pi\text{–regular element } d \in R \text{ such that } a^n - d \in J. \]

- Any local ring satisfies $(*)$.
- $(*) \not\Rightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$. 
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒
$\pi$–regular ⇒ weakly $\pi$–regular

(∗) For any $a \in R$, there exist a positive integer $n$ and a
right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies (∗).
- (∗) $\nRightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$.
◊ $R$ is local.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

(∗) For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies (∗).
- (∗) $\not\Rightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$.
- $R$ is local.
- $R$ satisfies (∗).
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

$(\star)$ For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.

- Any local ring satisfies $(\star)$.
- $(\star) \nRightarrow$ right weakly $\pi$–regular:

Example: Let $D$ be a division ring and $R = D[[x]]$.
- $R$ is local.
- $R$ satisfies $(\star)$.
- $R$ is not right weakly $\pi$–regular because $x^n \notin x^nRx^nR$ for all positive integer $n$. 

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1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.2**

R is called **semi right weakly $\pi$–regular rings**
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1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General

**Definition 7.2**

$R$ is called **semi right weakly $\pi$–regular rings** if $R/J$ is right weakly $\pi$–regular and idempotents lift modulo $J$. 
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.2**

R is called **semi right weakly $\pi$–regular rings** if $R/J$ is right weakly $\pi$–regular and idempotents lift modulo $J$.

**Theorem 7.3**

If $R$ satisfies ($\ast$), then $R/J$ is a right weakly $\pi$–regular ring.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

**Definition 7.2**

R is called semi right weakly $\pi$–regular rings if $R/J$ is right weakly $\pi$–regular and idempotents lift modulo $J$.

**Theorem 7.3**

If $R$ satisfies $(\star)$, then $R/J$ is a right weakly $\pi$–regular ring.

**Open Question**

If $R$ satisfies $(\star)$, then do idempotents lift modulo $J$?

((\star) For any $a \in R$, there exist a positive integer $n$ and a right weakly $\pi$–regular element $d \in R$ such that $a^n - d \in J$.)
If $R$ is right weakly $\pi$–regular, then $J$ is nil.
(Tuganbaev, 2002)
strongly regular \Rightarrow \text{strongly } \pi\text{-regular} \Rightarrow \text{unit } \pi\text{-regular} \Rightarrow \pi\text{-regular} \Rightarrow \text{weakly } \pi\text{-regular}

If $R$ is right weakly $\pi$-regular, then $J$ is nil. (Tuganbaev, 2002)

But if $R$ satisfies ($\ast$), then $J$ need not be nil.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

If $R$ is right weakly $\pi$–regular, then $J$ is nil. (Tuganbaev, 2002)

But if $R$ satisfies ($\ast$), then $J$ need not be nil. For example, the Jacobson radical of the local ring $\mathbb{Z}(p)$ is not nil.
**Strongly** regular $$\Rightarrow$$ **strongly** $$\pi$$–regular $$\Rightarrow$$ **unit** $$\pi$$–regular $$\Rightarrow$$
$$\pi$$–regular $$\Rightarrow$$ **weakly** $$\pi$$–regular

\[(**\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star\star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strongly regular \Rightarrow \text{strongly } \pi\text{–regular} \Rightarrow \text{unit } \pi\text{–regular} \Rightarrow \pi\text{–regular} \Rightarrow \text{weakly } \pi\text{–regular}

\text{(**)} For any \(a \in R\), there exist a positive integer \(n\) and \(x \in Ra^nR\) such that \(a^n - a^n x \in N^*(R)\).

**Proposition 7.4**

\(R/J\) is right weakly \(\pi\text{–regular} \) and \(J\) is nil \(\Leftrightarrow\)
strongly regular \Rightarrow \text{strongly } \pi\text{–regular } \Rightarrow \text{unit } \pi\text{–regular } \Rightarrow \\ \pi\text{–regular } \Rightarrow \text{weakly } \pi\text{–regular}

(\ast\ast) \text{ For any } a \in R, \text{ there exist a positive integer } n \text{ and } \\
x \in Ra^n R \text{ such that } a^n - a^n x \in N^*(R).

Proposition 7.4

$R/J$ is right weakly $\pi$–regular and $J$ is nil $\iff R$ satisfies (\ast\ast).
strongly regular \implies \text{strongly }\pi\text{–regular} \implies \text{unit }\pi\text{–regular} \implies \pi\text{–regular} \implies \text{weakly }\pi\text{–regular}

(∗∗) For any $a \in R$, there exist a positive integer $n$ and $x \in Ra^nR$ such that $a^n - a^nx \in N^*(R)$.

**Proposition 7.4**

$R/J$ is right weakly $\pi$–regular and $J$ is nil $\iff R$ satisfies (∗∗).

**Definition 7.5**

A ring $R$ is called right weakly regular if $(aR)^2 = aR$ for every $a \in R$ (Ramamurthi, 1973).
strongly regular ⇒ strongly $\pi$–regular ⇒ unit $\pi$–regular ⇒ $\pi$–regular ⇒ weakly $\pi$–regular

\((**)\) For any \(a \in R\), there exist a positive integer \(n\) and \(x \in Ra^n R\) such that \(a^n - a^n x \in N^*(R)\).

**Proposition 7.4**

\(R/J\) is right weakly $\pi$–regular and \(J\) is nil ⇔ \(R\) satisfies (**) .

**Definition 7.5**

A ring \(R\) is called right weakly regular if \((aR)^2 = aR\) for every \(a \in R\) (Ramamurthi, 1973).

**Proposition 7.6**

\(R/J\) is right weakly regular and \(J\) is nil ⇔
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
\[\pi\text{–regular} \Rightarrow \text{weakly } \pi\text{–regular}\]

(**) For any $a \in R$, there exist a positive integer $n$ and $x \in Ra^nR$ such that $a^n - a^n x \in N^*(R)$.

Proposition 7.4

$R/J$ is right weakly $\pi$–regular and $J$ is nil $\iff R$ satisfies (**).

Definition 7.5

A ring $R$ is called right weakly regular if $(aR)^2 = aR$ for every $a \in R$ (Ramamurthi, 1973).

Proposition 7.6

$R/J$ is right weakly regular and $J$ is nil $\iff$ for any $a \in R$, there exists $x \in RaR$ such that $a - ax \in N^*(R)$. 
Theorem 8.1

TFAE for an abelian ring $R$:
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TFAE for an abelian ring $R$:

1. $R$ is semi strongly $\pi$–regular.
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TFAE for an abelian ring $R$:

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Theorem 8.1

TFAE for an abelian ring $R$:

1. $R$ is semi strongly $\pi$–regular.
2. $R$ is semiregular.
3. $R$ is semi strongly regular.
4. $R$ is semi unit–regular.
Theorem 8.1

TFAE for an abelian ring $R$:

1. $R$ is semi strongly $\pi$–regular.
2. $R$ is semi regular.
3. $R$ is semi strongly regular.
4. $R$ is semi unit–regular.
5. $R$ is semi one–sided unit–regular.
Theorem 8.1

TFAE for an abelian ring $R$:

1. $R$ is semi strongly $\pi$–regular.
2. $R$ is semiregular.
3. $R$ is semi strongly regular.
4. $R$ is semi unit–regular.
5. $R$ is semi one–sided unit–regular.
6. $R$ is semi unit $\pi$–regular.
Theorem 8.1

TFAE for an abelian ring $R$:

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4. $R$ is semi unit–regular.
5. $R$ is semi one–sided unit–regular.
6. $R$ is semi unit $\pi$–regular.
7. $R$ is semi $\pi$–regular.
**Theorem 8.2**

TFAE for a right quasi-duo ring $R$:

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi unit–regular
5. Semi one–sided unit–regular
6. Semi unit $\pi$–regular
7. Semi $\pi$–regular
8. Semi right weakly $\pi$–regular
9. General
Theorem 8.2
TFAE for a right quasi-duo ring $R$:

1. $R$ is semi strongly $\pi$–regular.
2. $R$ is semiregular.
3. $R$ is semi strongly regular.
4. $R$ is semi unit–regular.
5. $R$ is semi one–sided unit–regular.
6. $R$ is semi unit $\pi$–regular.
7. $R$ is semi $\pi$–regular.
8. General
Theorem 8.2

TFAE for a right quasi-duo ring $R$:

1. $R$ is semi strongly $\pi$–regular.
2. $R$ is semiregular.
3. $R$ is semi strongly regular.
4. $R$ is semi unit–regular.
5. $R$ is semi one–sided unit–regular.
6. $R$ is semi unit $\pi$–regular.
7. $R$ is semi $\pi$–regular.
8. $R$ is semi right weakly $\pi$–regular.
Example 8.3

Let $D$ be a simple domain that is not a division ring.
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Let $D$ be a simple domain that is not a division ring. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$
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Let $D$ be a simple domain that is not a division ring. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

$\blacklozenge$ $R$ is right weakly $\pi$–regular (Hong, Kim, Kwak, Lee, 2000).
Example 8.3

Let $D$ be a simple domain that is not a division ring. Consider

$$R = \{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) | a, b \in D \}$$

- $R$ is right weakly $\pi$–regular (Hong, Kim, Kwak, Lee, 2000).
- $R/J$ is right weakly $\pi$–regular.
- $R/J$ is not $\pi$–regular: For a non–zero non–unit $a$ in $D$, $(a \ 0 \ 0 \ a) + J$ is not $\pi$–regular.
- $R$ is not right quasi–duo (Hong, Kim, Kwak, Lee, 2000).
Example 8.3

Let \( D \) be a simple domain that is not a division ring. Consider

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}
\]

- \( R \) is right weakly \( \pi \)-regular (Hong, Kim, Kwak, Lee, 2000).
- \( R/J \) is right weakly \( \pi \)-regular.
- \( J = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \) is nil.
Example 8.3

Let $D$ be a simple domain that is not a division ring. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

$\diamond$ $R$ is right weakly $\pi$–regular (Hong, Kim, Kwak, Lee, 2000).

$\diamond$ $R/J$ is right weakly $\pi$–regular.

$\diamond$ $J = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ is nil.

$\diamond$ $R$ is semi right weakly $\pi$–regular.
Example 8.3

Let $D$ be a simple domain that is not a division ring. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

- $R$ is right weakly $\pi$–regular (Hong, Kim, Kwak, Lee, 2000).
- $R/J$ is right weakly $\pi$–regular.
- $J = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ is nil.
- $R$ is semi right weakly $\pi$–regular.
- $R/J$ is not $\pi$–regular: For a non–zero non–unit $a$ in $D$, 
  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + J$ is not $\pi$–regular.
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Let $D$ be a simple domain that is not a division ring. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

- $R$ is right weakly $\pi$–regular (Hong, Kim, Kwak, Lee, 2000).
- $R/J$ is right weakly $\pi$–regular.
- $J = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$ is nil.
- $R$ is semi right weakly $\pi$–regular.
- $R/J$ is not $\pi$–regular: For a non–zero non–unit $a$ in $D$, $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + J$ is not $\pi$–regular.
- $R$ is not right quasi–duo (Hong, Kim, Kwak, Lee, 2000).
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Exchange Rings (Warfield, 1972)

1. Semiregular Rings

2. Semi (one–sided) unit–regular

3. Semi strongly regular

4. Semi $\pi$–regular

5. Semi unit $\pi$–regular

6. Semi strongly $\pi$–regular

7. Semi right weakly $\pi$–regular

8. General
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Exchange Rings (Warfield, 1972)

TFAE for a ring $R$:

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
Rings Closed to Semiregular

AYDOĞDU, ÖZCAN

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi \(\pi\)–regular
5. Semi unit \(\pi\)–regular
6. Semi strongly \(\pi\)–regular
7. Semi right weakly \(\pi\)–regular
8. General

Exchange Rings (Warfield, 1972)

TFAE for a ring \(R\):

1) For any \(a \in R\), there is an idempotent \(e \in aR\) such that \(1 - e \in (1 - a)R\).

strongly regular \(\Rightarrow\) strongly \(\pi\)–regular \(\Rightarrow\) unit \(\pi\)–regular \(\Rightarrow\) \(\pi\)–regular \(\Rightarrow\) weakly \(\pi\)–regular
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$

$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

#### Exchange Rings (Warfield, 1972)

TFAE for a ring $R$:

1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$
$\pi$–regular $\Rightarrow$ weakly $\pi$–regular

---

**Exchange Rings (Warfield, 1972)**

TFAE for a ring $R$:

1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.

3) All idempotents of $R$ can be lifted modulo every right ideal.

(Nicholson, 1977)
strongly regular \Rightarrow strongly \ \pi-regular \Rightarrow unit \ \pi-regular \Rightarrow \pi-regular \Rightarrow weakly \ \pi-regular

Exchange Rings (Warfield, 1972)

TFAE for a ring $R$:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of $R$ can be lifted modulo every right ideal.
(Nicholson, 1977)

\diamond Local rings, regular, semiregular rings are exchange.

\begin{itemize}
\item 1. Semiregular Rings
\item 2. Semi (one-sided) unit-regular
\item 3. Semi strongly regular
\item 4. Semi $\pi$-regular
\item 5. Semi unit $\pi$-regular
\item 6. Semi strongly $\pi$-regular
\item 7. Semi right weakly $\pi$-regular
\item 8. General
\end{itemize}
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Exchange Rings (Warfield, 1972)

TFAE for a ring $R$:
1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.
2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.
3) All idempotents of $R$ can be lifted modulo every right ideal.
   (Nicholson, 1977)

- Local rings, regular, semiregular rings are exchange.
- $\pi$–regular rings are exchange (Stock, 1986).
strongly regular $\Rightarrow$ strongly $\pi$–regular $\Rightarrow$ unit $\pi$–regular $\Rightarrow$ $\pi$–regular $\Rightarrow$ weakly $\pi$–regular

Exchange Rings (Warfield, 1972)

TFAE for a ring $R$:

1) For any $a \in R$, there is an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

2) For any $a \in R$, there is an idempotent $e \in R$ such that $e - a \in (a - a^2)R$.

3) All idempotents of $R$ can be lifted modulo every right ideal.

(Nicholson, 1977)

- Local rings, regular, semiregular rings are exchange.
- $\pi$–regular rings are exchange (Stock, 1986).
- Semi $\pi$–regular rings are exchange (Nicholson, 1977; Stock, 1986).

1. Semiregular Rings
2. Semi (one–sided) unit–regular
3. Semi strongly regular
4. Semi $\pi$–regular
5. Semi unit $\pi$–regular
6. Semi strongly $\pi$–regular
7. Semi right weakly $\pi$–regular
8. General
semigroup weakly $\pi$-regular $\not\iff$ exchange

Example 8.4

Let $D$ be a simple domain that is not a division ring. $R = \{ (a \ b \ 0 \ 0) | a, b \in D \}$.

$R$ is semi right weakly $\pi$-regular.

$R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).

$R = R/J$ is not exchange:

Let $0 \neq a \in D$ such that $a$ and $1 - a$ are not right unit.

Suppose $R$ is exchange. Then there exists an idempotent $e \in kR$ such that $(1 - e) \in (1 - k)R$, where $k = (a \ 0 \ 0 \ a) + J$.

Note that since $R \simeq D$, the only idempotents in $R$ are 0 and 1.

If $e = 0$, then $1 \in (1 - k)R$ so $(1 - k)R = R$. Let $y \in R$ such that $(1 - k)y = 1$.

This gives that there exists an element $b \in D$ such that $(1 - a)b = 1$, which is a contradiction.

If $e = 1$, then $a$ is a right unit in a similar way.

$R$ is not exchange.
Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \}$$
Example 8.4

Let $D$ be a simple domain that is not a division ring.

\[ R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\} \]

\[ R \text{ is semi right weakly } \pi\text{-regular.} \]
Rings Closed to Semiregular

AYDOĞDU, ÖZCAN

1. Semiregular Rings
2. Semi (one−sided) unit−regular
3. Semi strongly regular
4. Semi $\pi$−regular
5. Semi unit $\pi$−regular
6. Semi strongly $\pi$−regular
7. Semi right weakly $\pi$−regular
8. General

Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$ R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \} $$

◊ $R$ is semi right weakly $\pi$−regular.

◊ $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).
**Example 8.4**

Let $D$ be a simple domain that is not a division ring.

$$R = \{(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in D\}$$

- $R$ is semi right weakly $\pi$–regular.
- $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).
- $\overline{R} = R/J$ is not exchange:
Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

- $R$ is semi right weakly $\pi$–regular.
- $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).
- $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that $a$ and $1 - a$ are not right unit.
Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \{(a b) \mid a, b \in D\}$$

- $R$ is semi right weakly $\pi$–regular.
- $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).
- $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that $a$ and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange.
Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \}$$

$\diamond$ $R$ is semi right weakly $\pi$–regular.

$\diamond$ $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).

$\diamond$ $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that $a$ and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{kR}$ such that $(\overline{1} - \overline{e}) \in (\overline{1} - \overline{k})\overline{R}$, where $\overline{k} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + J$. 
Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}$$

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Example 8.4

Let $D$ be a simple domain that is not a division ring.

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Example 8.4

Let $D$ be a simple domain that is not a division ring.

$$R = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) \mid a, b \in D \right\}$$

- $R$ is semi right weakly $\pi$–regular.
- $R$ is exchange iff $R/J$ is exchange and idempotents lift modulo $J$ (Nicholson, 1977).
- $\overline{R} = R/J$ is not exchange: Let $0 \neq a \in D$ such that $a$ and $1 - a$ are not right unit. Suppose $\overline{R}$ is exchange. Then there exists an idempotent $\overline{e} \in \overline{kR}$ such that $\overline{(1 - e)} \in \overline{(1 - k)}\overline{R}$, where $\overline{k} = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) + J$. Note that since $\overline{R} \cong D$, the only idempotents in $\overline{R}$ are $\overline{0}$ and $\overline{1}$. If $\overline{e} = \overline{0}$, then $\overline{1} \in \overline{(1 - k)}\overline{R}$ so $\overline{(1 - k)}\overline{R} = \overline{R}$. Let $\overline{y} \in \overline{R}$ such that $\overline{(1 - k)}\overline{y} = \overline{1}$. This gives that there exists an element $b \in D$ such that $(1 - a)b = 1$, which is a contradiction. If $\overline{e} = \overline{1}$, then $a$ is a right unit in a similar way.
- $R$ is not exchange.
References

(Ara, 1996) Strongly $\pi$–regular rings have stable range one, Amer. Math. Soc.


References

(Goodearl-Menal, 1988) Stable range one for rings with many units, J. Pure Appl. Algebra.
(Kaplansky, 1951) Semi-simple alternative rings, Portugaliae Math.
(Stock, 1986) On rings whose projective modules have the exchange property, J. Algebra.
(Wu, 1994) Weak cancellation of modules and the weak stable range one condition, Nanjing Daxue Xuebao Shuxue Bannian Kan.
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